

# On The Iversen-Tsuji Theorem

JOHN B. GARNETT

*Communicated by the Editors*

1. Let  $D$  be an open set in the complex plane, let  $z_0 \in \partial D$ , and let  $f(\zeta)$  be a function defined on  $D$ . The cluster set of  $f$  at  $z_0$  is

$$\text{Cl}(f, z_0) = \bigcap_{r>0} \overline{f(D \cap \{|\zeta - z_0| < r\})}.$$

We write  $\sup |S|$  for  $\sup \{w : w \in S\}$ . The Iversen-Tsuji theorem [4], [6] states:

**Theorem 1.** *Let  $E$  be a compact subset of  $\partial D$  with capacity zero. Let  $z_0 \in \overline{(\partial D) \setminus E}$ ,  $z_0 \in E$ . Then whatever  $f(z)$  is analytic on  $D$*

$$(1.1) \quad \sup |\text{Cl}(f, z_0)| \leq \lim_{\substack{z \rightarrow z_0 \\ z \in (\partial D) \setminus E}} \sup |\text{Cl}(f, z)|.$$

Equivalent formulations of this theorem can be found in [4 p. 17].

Recently two new proofs of Theorem 1 have been given, by Gamelin [1] and by Martio and Rickman [3]. The proof in [3] uses path families and yields a more general theorem on quasiregular mappings in  $\mathbf{R}^n$ . The proof in [1] uses some special properties, from [2], of the Banach algebra  $H^\infty(D)$ . It yields a refinement of the theorem in which (1.1) holds when each  $\text{Cl}(f, z)$ ,  $z \in (\partial D) \setminus E$  is replaced by a smaller cluster set determined by restricting the approach to  $z$  from within  $D$ . It also yields a similar theorem for product domains in  $\mathbf{C}^n$ .

We give a direct proof of a theorem containing Theorem 1 and related results for [1]. This research arose from our trying to simplify the work in [1] and our debt to that paper is large.

Theorem 1 can be viewed as a local version (at a boundary point) of the maximum principle. Our proof uses Vitushkin's localization operator ([7], [2]) and the subharmonicity of  $\log |f(\zeta)|$ .

The notation  $f \in H^\infty(D)$  means that  $f$  is a bounded analytic function on  $D$ . Clearly it suffices to prove Theorem 1 for  $f \in H^\infty(D)$ . It also suffices to assume  $z_0 \in \partial D$  is an *essential boundary point*, which means by definition that not all  $f \in H^\infty(D)$  extend analytically to a neighborhood of  $z_0$ . We write  $\|f\|$  for  $\sup |f(\zeta)|$ , and  $\Delta(z, \delta)$  for the disc  $\{|\zeta - z| < \delta\}$ . We write  $\sup_{z \in S} |\text{Cl}(f, z)|$  for  $\sup_{z \in S} \sup \{|w| : w \in \text{Cl}(f, z)\}$ , where  $\sup \emptyset = -\infty$ .