

# Subspace Approximants and GCR Operators

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**§1. Introduction and notation.** In this paper we introduce the concept of subspace approximant and prove several results concerning subspace approximants. We also study some problems concerning GCR operators. All operators will be bounded linear operators on complex Hilbert spaces. We denote all operators from Hilbert space  $\mathcal{H}$  to Hilbert space  $\mathcal{H}'$  by  $\mathcal{B}(\mathcal{H}, \mathcal{H}')$ , while  $\mathcal{B}(\mathcal{H})$  will denote all operators from  $\mathcal{H}$  to  $\mathcal{H}$ . We denote by  $\mathcal{K}(\mathcal{H})$  the ideal of all compact operators on  $\mathcal{H}$ . For  $A \in \mathcal{B}(\mathcal{H})$ ,  $C^*(A)$  will denote the  $C^*$ -algebra generated by  $A$ ; that is, the smallest  $C^*$ -algebra containing  $A$  and  $I$ .

A  $C^*$ -algebra  $\mathcal{A}$  is called CCR if for every irreducible representation  $\pi$  of  $\mathcal{A}$  on a Hilbert space, and for every  $A \in \mathcal{A}$ ,  $\pi(A)$  is compact [7, 4.2.1]. A  $C^*$ -algebra  $\mathcal{A}$  is called GCR if  $\mathcal{A}$  has an increasing family of closed two-sided ideals  $(\mathcal{I}_\rho)_{0 \leq \rho \leq \alpha}$  satisfying,  $\mathcal{I}_0 = \{0\}$ ,  $\mathcal{I}_\alpha = \mathcal{A}$ ; if  $\rho \leq \alpha$  is a limit ordinal, then  $\mathcal{I}_\rho$  is the uniform closure of  $\bigcup \{\mathcal{I}_{\rho'} : \rho' < \rho\}$  and  $\mathcal{I}_{\rho+1}/\mathcal{I}_\rho$  is CCR [7, 4.3.4]. Equivalently,  $\mathcal{A}$  is GCR if every irreducible representation of  $\mathcal{A}$  contains a nonzero compact operator [15, 4.6.4]. An operator  $A \in \mathcal{B}(\mathcal{H})$  is called GCR if  $C^*(A)$  is a GCR  $C^*$ -algebra. We would like to thank W. Arveson for several stimulating conversations.

**§2. Subspace approximants.** We say that  $F \in \mathcal{B}(\mathcal{H}')$  is a *subspace approximant* to  $A \in \mathcal{B}(\mathcal{H})$  if for every  $\epsilon > 0$  there is an operator  $R_\epsilon \in \mathcal{B}(\mathcal{H}_\epsilon)$  and a unitary operator  $U_\epsilon \in \mathcal{B}(\mathcal{H}, \mathcal{H}' \oplus \mathcal{H}_\epsilon)$  such that

$$\|A - U_\epsilon^*(F \oplus R_\epsilon)U_\epsilon\| < \epsilon.$$

Here we also allow the possibility of  $R_\epsilon$  being absent in which case we require that  $U_\epsilon \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$  and that  $\|A - U_\epsilon^*FU_\epsilon\| < \epsilon$ . If for all  $\epsilon > 0$  there is an unitary  $U_\epsilon$  with  $\|A - U_\epsilon^*FU_\epsilon\| < \epsilon$ , we say that  $F$  is in the closure  $\bar{\mathcal{U}}_A$  of the unitary equivalence class of  $A$ . We note that  $F \in \mathcal{B}(\mathcal{H}')$  is a subspace approximant for  $A$  if and only if for all  $\epsilon > 0$  there is an isometry  $V_\epsilon : \mathcal{H}' \rightarrow \mathcal{H}$  such that  $\|V_\epsilon^*AV_\epsilon - F\| < \epsilon$  and  $\|V_\epsilon V_\epsilon^*A - AV_\epsilon V_\epsilon^*\| < \epsilon$ .