

Extreme Points and Finite Rank Operators in the Set of Compact Approximants

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Let T be a bounded linear operator on a Hilbert space \mathcal{H} . A *compact approximant* of T is a compact operator K of minimum distance from T ; that is,

$$\|T - K\| = \inf \{\|T - C\| : C \text{ compact}\}.$$

We call this distance μ , the *essential norm* of T [8]. The set \mathcal{K}_T of compact approximants is a closed convex subset of the sphere of radius μ and center T in $\mathcal{B}(\mathcal{H})$. In this note we explicitly describe the extreme points of \mathcal{K}_T , and exhibit the structure of \mathcal{K}_T in terms of these extreme points. There may be no extreme points in \mathcal{K}_T ; however if there is one, then every element of \mathcal{K}_T is a midpoint between two extreme points. Similarly (but under unrelated conditions) there may be no finite rank operators in \mathcal{K}_T ; but if there is one, then they are uniformly dense in \mathcal{K}_T .

These results contribute to the continuing study of non-commutative approximation theory initiated by P. R. Halmos [5]. Also, they focus on the Banach space structure of $\mathcal{B}(\mathcal{H})$ by examining the position of the distinguished closed subspace of compact operators. It was originally shown by I. Gohberg and M. G. Krein in [3] that a compact approximant exists for every $T \in \mathcal{B}(\mathcal{H})$. More general existence theorems are given in [1] concerning M -ideals in Banach spaces, and in [9] for closed two-sided ideals in C^* -algebras. Compact approximants and related topics have also been studied in [6], [7], and [8].

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We write $\mathcal{B}(\mathcal{H})$ for the algebra of all bounded linear operators on a separable Hilbert space \mathcal{H} , and \mathcal{K} for the closed ideal in $\mathcal{B}(\mathcal{H})$ of compact operators. The map $\nu: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})/\mathcal{K}$ is the natural homomorphism of $\mathcal{B}(\mathcal{H})$ onto the C^* -algebra quotient, the *Calkin algebra*. Let $\sigma(T)$ and $\sigma(\nu(T))$ be the spectra of $T \in \mathcal{B}(\mathcal{H})$ and of $\nu(T)$ in the Calkin algebra, respectively. The operator $(T^*T)^{1/2}$