

Schauder Bases and Norm Ideals of Compact Operators II

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Communicated by the Editors

§1. In an earlier paper [6] we showed how problems concerning norm ideals of compact operators on a Hilbert space can often be solved quite easily if one first translates the problem to an equivalent one concerning the properties of a Schauder basis in a suitable sequence space. The purpose of this paper is to develop further the ideas and techniques introduced in [6]. In particular we solve completely the problem of “intermediate” ideals first studied by Mitiagin [8].

The origin of the problem is as follows: Let \mathfrak{H} denote a separable infinite dimensional Hilbert space (we assume \mathfrak{H} is a real Hilbert space but the results hold, as usual, in the complex case). Let $B(\mathfrak{H})$ denote the ring of all bounded linear operators on \mathfrak{H} and $K(\mathfrak{H})$ the ideal in $B(\mathfrak{H})$ consisting of all compact operators. A two-sided ideal J in $B(\mathfrak{H})$ is called a *norm ideal* if there is a norm α defined on J for which

- (i) J is complete under α .
- (ii) $\alpha(T) = \|T\|$ if T is a one dimensional operator.
- (iii) $\alpha(U \circ T \circ V) \leq \|U\| \|V\| \alpha(T)$ for all $T \in J$ and all $U, V \in B(\mathfrak{H})$.

The norm ideal J is called *minimal* if it contains no proper closed subspace which is a norm ideal in $B(\mathfrak{H})$, and *maximal* if it is not itself a closed proper subspace of another norm ideal. If J is a maximal norm ideal, then it contains a unique minimal norm ideal \dot{J} , namely the closure of the finite dimensional operators on \mathfrak{H} .

The simplest example of such a pair of norm ideals is the maximal ideal $B(\mathfrak{H})$ and its minimal ideal $K(\mathfrak{H})$. Now Calkin has shown [2] that there is no norm ideal (in fact, no two-sided ideal) properly contained between $K(\mathfrak{H})$ and $B(\mathfrak{H})$. At one time it was unknown whether this situation is characteristic of all pairs of minimal—maximal norm ideals. That this is not the case was shown by Mitiagin [8] who constructed a norm ideal properly contained between the minimal ideal \dot{S}_π and the maximal ideal S_π of Gohberg and Krein [5]. The general problem of characterizing those pairs of minimal—maximal norm ideals which admit no intermediate norm ideals has remained open.