

# Existence of Hyperinvariant Subspaces

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**Introduction.** Let  $\mathfrak{X}$  be a complex Banach space, and let  $S$  be a bounded linear operator on  $\mathfrak{X}$ . A subspace (*i.e.* a closed linear manifold)  $\mathfrak{M}$  of  $\mathfrak{X}$  is *invariant* under  $S$  if  $S\mathfrak{M} \subset \mathfrak{M}$ .  $\mathfrak{M}$  is a hyperinvariant subspace of  $S$  if it is invariant under any operator commuting with  $S$ . A well known unsolved problem is the invariant subspace problem: whether or not a bounded linear operator on a complex Banach space has a non-trivial invariant subspace. Another related unsolved problem is: whether or not a bounded linear operator which is not a multiple of the identity on a complex Banach space has a non-trivial hyperinvariant subspace. It is clear that a positive answer to the second problem will give a positive answer to the first one. While both problems are still unsolved, in quite a few cases (scalar operators, operators commuting with a compact operator, operators with some sort of growth condition on their resolvent, *etc.*) the answer is known to be in the affirmative.

Let  $T$  be a bounded linear operator on a complex Hilbert space  $\mathfrak{H}$ . Macaev and Gohberg–Krein (*cf.* Macaev [10]), and also Schwartz [15], generalizing earlier results of Livsic [9] and Sahnovic [14] proved: if  $T = A + iB$ , where  $A = (T + T^*)/2$ ,  $B = (T - T^*)/2i$ , and  $B \in \mathfrak{C}_p$  (a Von Neumann–Schatten class), then  $T$  has a non-trivial invariant subspace. In 1967, K. Kitano [8] improved this result to the case  $T = A + B$  where  $B \in \mathfrak{C}_p$  and  $A$  is a normal operator with spectrum on a  $C^2$ -Jordan arc. H. Radjavi and P. Rosenthal [13] simplified Kitano's proof and showed that it is enough that  $\sigma(A)$  or  $\sigma(T)$  contains an exposed  $C^2$ -Jordan arc. In this paper we generalize these results. We prove that if  $T = A + B$  where  $B \in \mathfrak{C}_p$ ,  $\sigma(A)$  or  $\sigma(T)$  contains an exposed  $C^2$ -Jordan arc  $J$ , and  $\|(\lambda - A)^{-1}\| \leq K [\text{dist}(\lambda, J)]^{-m}$ , then  $T$  has a non-trivial hyperinvariant subspace. Several corollaries are also derived. Some of our results are closely related to earlier results of C. Apostol. We only became aware of Apostol's work after ours had been completed. Our proofs are different from those of [1] and are more similar to those of [8], [13], [15].

**§1. Preliminaries.** In this section we quote some definitions, theorems and lemmas which will be necessary for our work in section 2. Throughout the