

Gaussian Sums over $GL(2, \mathbf{Z}/N)$ and Hecke Operators I

ROY J. FULLER

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If R is a representation of $GL(2, \mathbf{Z}/N)$ and A is a matrix in $M(2, \mathbf{Z})$, then the Gaussian sum $W(A)$ is defined by

$$W(A) = \sum_g R(g) \exp [2\pi i \text{trace } (Ag)/N]$$

where g is summed over $GL(2, \mathbf{Z}/N)$. These Gaussian sums lead to a Hecke operator which acts on automorphic forms "of type R ," which appeared, in a more general setting, in Shimura's paper [2], where it was suggested that the associated Dirichlet series might have an Euler product. Our goal is to calculate the " p factor" of the Euler product for the prime divisors of N . The main result is Proposition 6.

1. We begin by reviewing some definitions from [2], making the appropriate specializations. R will denote an irreducible representation of $GL(2, \mathbf{Z}/N)$ by complex matrices of size d . We always suppose that R is primitive, *i.e.*, R is not obtained by composing a representation of $GL(2, \mathbf{Z}/M)$ with the natural homomorphism $GL(2, \mathbf{Z}/N) \rightarrow GL(2, \mathbf{Z}/M)$, where M is a proper divisor of N . Let $\mathbf{C}^+ = \{z \mid \text{Im}(z) > 0\}$ denote the upper half of the complex plane, and let \mathbf{C}^d denote the space of complex column vectors of dimension d . For each function

$$f : \mathbf{C}^+ \rightarrow \mathbf{C}^d \quad \text{and each } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ in } M(2, \mathbf{Z})$$

such that $\det(A) > 0$, we have the A -transform $f \mid A$ of weight k :

$$(f \mid A)(z) = \det(A)^{k/2} (cz + d)^{-k} f((az + b)/(cz + d)).$$

f is a *cuspidal form of type (R, k)* if the following conditions are satisfied:

(1) The coordinate functions of f are all cusp forms of weight k with respect to the principal congruence subgroup $\Gamma_N = \{A \in \Gamma \mid A \equiv I \pmod{N}\}$.

(2) $(f \mid A)(z) = R(A)f(z)$ for all A in $\Gamma = SL(2, \mathbf{Z})$.

$S(R, k)$ will denote the space of cusp forms of type (R, k) .