

M_3 Functions

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The class M_1 of Baire 1 functions having the Darboux property is the subject of much research. (See for example [1].) This is partially due to the fact that M_1 contains several important types of functions, such as derivatives, approximate derivatives, Peano derivatives, and approximately continuous functions.

In his study [2] of derivatives, Zahorski defines a finite nested sequence of classes of functions of which M_1 is the largest and M_5 , the approximately continuous functions, the smallest. Intermediate is M_3 which he proves contains the derivatives. Further, C. Weil in [3] shows that approximate derivatives and Peano derivatives are in M_3 .

C. J. Neugebauer [4], meanwhile, presents two interesting characterizations of M_1 functions and one characterization of derivatives. These results are especially interesting in that they provide a good indication of how an M_1 function fails to be a derivative. Motivated by this work, in this paper we give necessary and sufficient conditions for a function to be in M_3 . Not only do these conditions provide a view of how M_3 functions differ from M_1 functions but, also, in a precise sense, how M_3 functions fail to be continuous. Finally, we use the results to get simple proofs that derivatives and approximate derivatives are M_3 functions. We will need several definitions. For completeness and comparison we include those of Neugebauer's paper.

Let I_0 be the interval $[0, 1]$ and \mathcal{J} the collection of all non-degenerate closed subintervals of I_0 . We take \mathcal{A} to be the collection of all sets A of the form $J \setminus Z$ where J belongs to \mathcal{J} and Z is of measure zero. Whenever we use the notation J_n or A_n we will require tacitly that J_n is in \mathcal{J} for every n , and A_n is in \mathcal{A} for every n .

Definition 1. A Baire 1 function $f: I_0 \rightarrow \mathbf{R}$ is an M_3 function if for every x_0 and for every $\epsilon > 0$ we have $m(A_n)/d(x_0, A_n) \rightarrow 0$ for every sequence A_n such that i) $m(A_n) \rightarrow 0$ and ii) $A_n \cap \{x: d(f(x), f(x_0)) < \epsilon\} = \emptyset$ for every n . Here m denotes Lebesgue measure and d the distance function.

Definition 2. A sequence J_n is said to converge to x_0 , denoted by $J_n \rightarrow x_0$, if i) $m(J_n) \rightarrow 0$ and ii) x_0 is in J_n for every n .