

Transience, Density, and Point Recurrence of Multiparameter Brownian Motion

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The transience, density, and point recurrence properties of one-parameter Brownian motion have been known for some time. If the Brownian path is in three-dimensional Euclidean space, the path is nowhere dense with probability one and goes to infinity as the time parameter goes to infinity. In d -dimensional space, $d \geq 5$, the probability is one that the path has no double points [2]. (A different proof [1] shows there are no double points in 4-space.)

The original proofs of these properties depend on Lévy's equality:

$$(1) \quad P\left\{\max_{t \in [a, b]} X(t, \omega) - X(a, \omega) > \lambda\right\} = 2P\{X(b, \omega) - X(a, \omega) > \lambda\}.$$

This equality requires independent increments, which do not exist in general for multiparameter Brownian motion. It turns out that Lévy's equality is not needed. The non-recurrence property follows from Lévy's result on the modulus of continuity of multiparameter Brownian motion [3]. The transience and density properties follow from an inequality given here as Theorem 2. These proofs seem to be new, even for the one-parameter case.

Let $W^{(N, d)}$ denote Lévy's N -parameter Brownian motion with values in d -dimensional Euclidean space; i.e. if $X = W^{(N, d)}$, then $X(t, \omega) = (X_1(t, \omega), \dots, X_d(t, \omega)) \in \mathbf{R}^d$, where $t = (t_1, \dots, t_N) \in \mathbf{R}^N$ and the coordinate functions X_i are mutually independent, separable, Gaussian processes with mean zero and covariance

$$E(X_i(s), X_i(t)) = (1/2)[|s| + |t| - |s - t|].$$

Here $|\cdot|$ is the Euclidean norm.

Theorem 1. *If $4N < d$, then almost no paths of $W^{(N, d)}$ have double points; i.e. $P\{X(s, \omega) = X(t, \omega) \text{ for some distinct } s, t\} = 0$.*

Proof. We use Lévy's modulus of continuity [3], which shows that if $0 < \alpha < 1/2$ and if A is a bounded set in \mathbf{R}^N , then