

Cardinal Spline Interpolation Operators on l^p Data

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Introduction. For a natural number n , the space $\mathcal{S}_n = \{S(x)\}$ of bounded cardinal splines of degree $n - 1$ is taken to consist of those functions satisfying:

- (i) $S \in C^{n-2}(-\infty, +\infty)$
- (ii) $\|S\| = \sup_{-\infty < x < +\infty} |S(x)| < +\infty$
- (iii) $S(x)$ reduces to a polynomial of degree at most $n - 1$ on each of the intervals $[\nu + \frac{1}{2}n, \nu + \frac{1}{2}n + 1]$, $\nu = 0, \pm 1, \pm 2, \dots$, i.e. $S(x)$ has knots at the integers or half-integers if n is respectively even or odd.

For a sequence $y = \{y_\nu\}_{\nu=-\infty}^{+\infty} \in \ell^p(\mathbf{Z})$, the Banach space of doubly infinite bounded sequences with the usual norm $\|y\|_{\ell^p} = \sum_{\nu=-\infty}^{+\infty} |y_\nu|^p$, there is a unique element $\mathcal{L}_n y \in \mathcal{S}_n \cap L^p(\mathbf{R})$ interpolating the given data at the integers, i.e. $\mathcal{L}_n y(\nu) = y_\nu$, $\nu \in \mathbf{Z}$ (Schoenberg [9]). The operator $\mathcal{L}_n : \ell^p(\mathbf{Z}) \rightarrow \mathcal{S}_n \cap L^p(\mathbf{R})$ is called the cardinal spline interpolation operator of order n , and its norm is

$$\|\mathcal{L}_n\|_p = \sup_{\|y\|_{\ell^p} \leq 1} \|\mathcal{L}_n y\|_{L^p}.$$

It is the purpose of the present paper to investigate the behaviour of $\|\mathcal{L}_n\|_p$, p -fixed, as n tends to infinity. For the case $p = +\infty$, Richards [5] has shown that $\|\mathcal{L}_n\|_\infty$ is asymptotic to $(2/\pi) \log n$; a result which is similar to the "best" rates obtainable for polynomial interpolation on $[-1, 1]$ (Rivlin [7] and Erdős [2]). For $p = 1$, in section 2 we show that $\|\mathcal{L}_n\|_1$ is asymptotic to $(4/\pi^2) \log n$ which is of the same order as the L^1 -norm of the Dirichlet kernel that relates to the "best" rate obtainable for projection operators from $C[-\pi, \pi]$ to trigonometric polynomials of degree n .

When $p = 2$, Richards [4], for n -even, and Marsden and Mureika [3], for n -odd, have shown that $\|\mathcal{L}_n\|_2 = 1$. The behaviour of $\|\mathcal{L}_n\|_p$ for the cases $p = 1, 2$, and $+\infty$ suggested the Hilbert transform. In section 3, we obtain the uniform boundedness of $\|\mathcal{L}_n\|_p$, $1 < p < +\infty$, in terms of the norm of a Hilbert trans-