

Homotopy Groups of Diffeomorphism Groups

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0. This paper contains some results on the homotopy type of the groups of orientation preserving diffeomorphisms of spheres, $\text{Diff } S^n$. C^∞ maps, manifolds, and the C^∞ topology on Diff will be used throughout. It is known [5] that the inclusion of $SO(n+1)$ in $\text{Diff } S^n$ induces a splitting $\pi_i(\text{Diff } S^n) \simeq \pi_i(SO(n+1)) \oplus \pi_i(\text{Diff}(D^n, S^{n-1}))$ for all i , where $\text{Diff}(D^n, S^{n-1})$ is the group of diffeomorphisms of the ball D^n fixing S^{n-1} . There are maps $\gamma_n^i : \pi_i(\text{Diff}(D^n, S^{n-1})) \rightarrow \pi_{i-1}(\text{Diff}(D^{n+1}, S^n))$ [16] and $\beta_n : \pi_0(\text{Diff}(D^n, S^{n-1})) \rightarrow \Gamma_{n+1}$, Γ_n being the group of concordance classes of differential structures on the PL n -sphere. β_n is defined by extending a representative diffeomorphism to S^n by the identity and using the resulting map to identify two copies of D^{n+1} . The definition of γ_n^i will be given later. β_n is known to be always onto and an isomorphism except possibly for $n = 4$ [6]. Define $\lambda_n^i = \beta_{i+n} \circ \gamma_{i+n-1}^1 \circ \cdots \circ \gamma_n^i : \pi_i(\text{Diff}(D^n, S^{n-1})) \rightarrow \Gamma_{i+n+1}$. The object is to investigate the image groups of the λ_n^i , Γ_{n+i+1}^i , defined by Gromoll.

The result of greatest interest, stated in §1 as Theorem 3, is an application of a theorem of Morlet [14; not yet published in final form]. It says, roughly speaking, that a differentiable homeomorphism $\Sigma^n \rightarrow S^n$, Σ^n an exotic sphere, of high rank and with nice singularities, gives rise to nontrivial elements of $\pi_i(\text{Diff}(D^{n-i-1}, S^{n-i-2}))$. §2 discusses a construction due to Bredon [3] which leads to the following applications of Theorem 3.

Let bP_{n+1} be the subgroup of Γ_n consisting of exotic spheres bounding smooth parallelizable manifolds. By Kervaire and Milnor [12] there is a monomorphism $\Gamma_n/bP_{n+1} \rightarrow \Pi_n/\text{Im } J_n$ onto a subgroup of index at most 2. Browder [4] showed it is onto except possibly when n is of the form $2^i - 2$. Π_n is the stable n -stem and $J_n : \pi_n(SO) \rightarrow \Pi_n$ is the stable J -homomorphism.

Theorem 1. $\Gamma_{n+i+1}^i/\Gamma_{n+i+1}^i \cap bP_{n+i+2}$ contains the subgroup generated by the images of the composition pairings

$$\Gamma_j \otimes \Pi_k \rightarrow \Pi_j/\text{Im } J_j \otimes \Pi_k \rightarrow \Pi_{n+i+1}/\text{Im } J_{n+i+1}$$

for $j+k = n+i+1$, $i+1 \leq k < j-1$.