

An Euler Characteristic Formula for Maps Between Manifolds

DONALD R. SHORT

Communicated by S. S. CHERN

1. Introduction. A unified approach to Euler characteristic formulas for maps between manifolds is developed via the Leray spectral sequence. For example, when the general formula developed here is restricted to fibre bundles it becomes $\chi(E) = \chi(B)\chi(F)$, or to branched coverings in Riemann surface theory it becomes the classical Riemann–Hurwitz formula.

Briefly, the development proceeds as follows: first a general Euler characteristic formula for finitely generated spectral sequences is derived. This formula is then applied to the Leray spectral sequence of the map $f: M \rightarrow N$ giving the general Euler characteristic formula for this map

$$\chi(M) - \chi(\partial M) = \sum_q (-1)^{q+m} \chi(H^*_c(N; \mathfrak{H}^q_c(f; R))).$$

Because of the occurrence of sheaf cohomology terms in this formula, a computational procedure is developed. Finally, using these techniques on three restricted classes of maps, the formula for these classes is expressed in terms of singular homology.

2. Euler characteristic of a spectral sequence. Suppose $A = \{A_i\}$, $B = \{B_i\}$ and $C = \{C_i\}$ are graded modules over a principal ideal domain connected by the long exact sequence

$$\cdots \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow A_{i+1} \rightarrow \cdots .$$

In this case the classical Euler characteristic theorem asserts that if any two of these graded modules is finitely generated, then so is the third and that the formula $\chi(B) = \chi(A) + \chi(C)$ is valid. It is well known that without further hypothesis this formula gives the best information about these modules. The first theorem of this paper is the analogous result for finitely generated spectral sequences.

Suppose $\{E^{p,q}_2\} \Rightarrow A^{p+q}$ is a spectral sequence of modules over a principal