

Chains of Primes in Noetherian Rings

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Introduction. All rings mentioned in this work are commutative Noetherian with one, unless otherwise stated. The paper consists of three parts. In the first section, the authors study chains of prime ideals in $R[x]$, where R is a Noetherian domain and x is an indeterminate. The main result of the first section is the following:

Theorem. *Let P be prime in a Noetherian domain R ; and in $R[x]$ let K be a nonzero prime with $K \cap R = 0$ and let Q be a prime with $Q \cap R = P$ but $Q \neq PR[x]$. If $K \subset Q$ and $\text{rank } Q/K < \text{rank } P$, then either $K \subset PR[x]$ or there is a prime P' in the integral closure of R such that $P' \cap R = P$ and $\text{rank } P' = \text{rank } Q/K$.*

A number of consequences of this result are obtained. For instance, if (R, M) is a local domain and Q is prime in $R[x]$ with $Q \cap R = M$ but $Q \neq MR[x]$, then it is shown that the following three sets are equal: $\{n - 1 \mid n \text{ is the length of some saturated chain of primes in } R[x] \text{ from } 0 \text{ to } Q\}$, $\{m \mid m \text{ is the length of some maximal chain of primes in some simple integral extension domain of } R\}$, and $\{m \mid m \text{ is the length of some maximal chain of primes in some integral extension domain of } R\}$. The second section applies arguments in the first section to the special case of local catenary domains. For example, if (R, M) is a local domain and Q is any prime in $R[x]$ with $Q \cap R = M$ but $Q \neq MR[x]$, then $R[x]$ is catenary if and only if $R[x]_Q$ is catenary. In the last section two examples and three conjectures are discussed. It is shown that the truth of any of the conjectures would imply that integral extension domains of local catenary domains are catenary.

1. Relations between R and $R[x]$.

Notation and Definitions. Let R be a not necessarily Noetherian domain and let x be an indeterminate. For a prime P of R , we will denote the prime $PR[x]$ by P^* . If Q is a prime of $R[x]$ with $Q \cap R = P$ but $Q \neq P^*$, then we shall call Q an *upper* to P .