

Resolvent Techniques and Quasinilpotent Operators

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By “resolvent techniques”, we will mean growth conditions on the resolvent function, $\lambda \rightarrow R(\lambda, A) = (\lambda - A)^{-1}$, which is defined and analytic on $\rho(A)$, the resolvent set of the operator A on a Hilbert space H . The purpose of this paper is to find a growth condition which will ensure the existence of proper invariant subspaces for quasinilpotent operators on H .

An operator A will be called a (G_n) operator if the inequality

$$(1) \quad \|R(\lambda, A)\| \leq K(\text{dist} [\lambda, \sigma(A)])^{-n}$$

is satisfied for some positive constant K , some non-negative integer n , and for all λ in $\rho(A)$. An operator is called locally (G_n) if this inequality is satisfied in some neighborhood, U , of $\sigma(A)$, the spectrum of A .

A weaker growth condition on A is given by the inequality

$$(2) \quad |(R(\lambda, A)x, y)| \leq K(\text{dist} [\lambda, \sigma(A)])^{-n}$$

where x and y are fixed vectors in H , and K and n are as above. These will be called (G_n, x, y) operators. The notion of locally (G_n, x, y) operators is defined analogously to the case for (G_n) operators above.

It is known that hyponormal operators are of type (G_1) , (cf. Putnam [3], p. 58), and it is easy to see that a (G_1) or (G_1, x, y) operator is locally (G_n) or locally (G_n, x, y) , respectively, for $n > 1$. Also, it is easy to show that any of the above growth conditions are invariant under translations and multiplication by non-zero scalars.

Note that, for the case of a quasinilpotent operator A , we have $\text{dist} [\lambda, \sigma(A)] = |\lambda|$.

It will be shown that inequality (2) characterizes those quasinilpotent operators with proper invariant subspaces, whereas inequality (1) is too strong for this purpose. All proofs are straightforward applications of the functional calculus and elementary complex variables, and will be omitted.