

# *On the Solutions of a Nonlinear Dirichlet Problem*

M. S. BERGER & E. PODOLAK

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In a recent paper [1], Ambrosetti and Prodi studied the structure of the solutions of the nonlinear Dirichlet problem

$$(1) \quad \Delta u + f(u) = g \quad u|_{\partial\Omega} = 0$$

for  $g$  in the Hölder space  $C^{0,\alpha}(\Omega)$  ( $0 < \alpha < 1$ ) defined over a smooth bounded domain  $\Omega \subset \mathbf{R}^N$ . Here  $\Delta$  is the Laplace operator and  $f(s)$  is a  $C^2$  strictly convex real valued function with  $f(0) = 0$  that is related to the eigenvalues  $(\lambda_1, \lambda_2, \dots)$  of  $\Delta$  (on  $\Omega$ ) by the facts:

$$(2) \quad (i) \quad \lambda_1 < \lim_{s \rightarrow \infty} f(s)/s < \lambda_2 \quad (ii) \quad 0 < \lim_{s \rightarrow -\infty} f(s)/s < \lambda_1 .$$

In [1], the following interesting facts concerning (1) were established by regarding the operator  $Au = \Delta u + f(u)$  as a differentiable mapping between the Hölder spaces  $C^{2,\alpha}(\bar{\Omega})$  and  $C^{0,\alpha}(\bar{\Omega})$ :

(A) The singular values of the mapping  $A$  (*i.e.*, the image under  $A$  of the points  $u$  at which the Frechet derivative  $A'(u)$  is not invertible) form a connected manifold  $M$  of codimension one in  $C^{0,\alpha}(\bar{\Omega})$  such that  $C^{0,\alpha}(\bar{\Omega}) - M$  has exactly two components  $O_0$  and  $O_2$ .

(B) For  $g \in O_0$ , the equation (1) has no solutions while for  $g \in O_2$ , (1) has exactly two solutions.

(C) Finally, if  $g \in M$ , (1) has exactly one solution.

However the techniques used in [1] are not adequate to determine necessary and sufficient conditions for  $g$  to be an element of either  $O_0$ ,  $O_2$  or  $M$ . In this paper we give an alternate simple Hilbert space approach to the study of (1) that (i) yields (A)–(C) for the more general case when  $A$  is regarded as a mapping between the Sobolev spaces  $\dot{W}_{1,2}(\Omega)$  and  $W_{-1,2}(\Omega)$  and (ii) provides sharp criteria for  $g$  to belong to one of the resulting sets  $M$ ,  $O_0$  or  $O_2$ . Immediate consequences of our results are:

(D) The number of solutions of (1) are determined by the size of the projection