

Spatial Homomorphisms of Operator Algebras

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1. Introduction. Suppose that \mathfrak{X} and \mathfrak{Y} are Banach spaces and \mathfrak{A} is a subalgebra of $\mathfrak{B}(\mathfrak{X})$, the algebra of all operators (bounded linear transformations) on \mathfrak{X} . A homomorphism φ of \mathfrak{A} into $\mathfrak{B}(\mathfrak{Y})$ is called a *spatial homomorphism* if there exist operators $S: \mathfrak{X} \rightarrow \mathfrak{Y}$ and $T: \mathfrak{Y} \rightarrow \mathfrak{X}$ such that $\varphi(A) = SAT$ for A in \mathfrak{A} . While many investigators have found occasion to use certain special spatial homomorphisms in the theory of operators on Hilbert space, the systematic study of these maps seems to have been neglected. It is the purpose of this paper to begin such a study.

It will be seen that there is an analogy between the theory of spatial homomorphisms and the theory of ring homomorphisms. In this analogy certain invariant subspaces play the role of ideals. For example, it will be shown that if the algebra \mathfrak{A} has no non-trivial invariant subspaces, then any non-zero spatial homomorphism on \mathfrak{A} is an isomorphism. (Such terms as spatial isomorphism and spatial endomorphism have their obvious meanings and will be used without further comment. It will also be convenient to speak of maps into operator algebras as being spatially homomorphic, similar, etc. Thus π is a spatial homomorphism of σ means that $\pi = \varphi \circ \sigma$, where φ is a spatial homomorphism.)

The examples that follow will serve to illustrate and motivate the definitions of some special spatial homomorphisms.

Example 1. Observe that for any operators that map $A \rightarrow SAT$ in linear, but not always multiplicative. However, if S is an isomorphism of \mathfrak{X} onto \mathfrak{Y} , then $A \rightarrow SAS^{-1}$ is a spatial homomorphism on every subalgebra of $\mathfrak{B}(\mathfrak{X})$. We shall keep to the convention that such a map is called a *similarity* only if the spaces \mathfrak{X} and \mathfrak{Y} are identical.

Example 2. The preceding example can be modified in the following way. Let \mathfrak{X}_0 and \mathfrak{Y}_0 be Banach spaces. Let \mathfrak{A} be a subalgebra of $\mathfrak{B}(\mathfrak{X} \oplus \mathfrak{X}_0)$ containing only operators of the form $A \oplus 0$, then the map φ defined by

$$\varphi(A \oplus 0) = SAS^{-1} \oplus 0 = (S \oplus 0)(A \oplus 0)(S^{-1} \oplus 0)$$