

# *Equivalence for Groups and Semigroups of Operators*

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**1. Introduction.** In [6] T. B. Hoover and the present author showed that if  $U$  and  $V$  are isometries on a Hilbert Space  $\mathcal{H}$  and  $\sup_n \|U^n - V^n\| < 1$ , then  $U$  and  $V$  are unitarily equivalent. This may be viewed as a theorem about "close" discrete isometric semigroups. The question arises as to whether there is an analogous result for continuous isometric semigroups. This is shown to be true (Theorem 3.5). The proof depends on Lemma 3.4, which seems likely to be of use in other investigations. Before proving these results we show that close unitary group representations of  $\sigma$ -compact abelian groups are unitarily equivalent. Our basic tool is the existence of a weak form of the wave operator developed in time-independent scattering theory ([7]). We show how this result can be used to investigate the unitary equivalence of unbounded normal and self adjoint operators.

**2. Preliminaries.** Let  $\mathcal{H}$  be a complex Hilbert space and let  $\mathcal{B}(\mathcal{H})$  be the algebra of all bounded linear operators on  $\mathcal{H}$ . Throughout this paper the term "operator" will be reserved for an everywhere defined, bounded linear transformation from  $\mathcal{H}$  into  $\mathcal{H}$ . In general, linear transformations will only be assumed to be defined on linear manifolds in  $\mathcal{H}$ . If  $T$  is a linear transformation, its domain will be denoted by  $\mathcal{D}(T)$ . Also the kernel, or null space of  $T$  will be denoted by  $\text{Ker}(T)$ . If  $\mathcal{G} \equiv (\mathcal{G}, +)$  is a topological group (or semigroup), by a strongly continuous representation of  $\mathcal{G}$  on  $\mathcal{H}$  we mean a mapping  $g \rightarrow A_g$  of  $\mathcal{G}$  into  $\mathcal{B}(\mathcal{H})$  such that

- (i)  $A_{g+h} = A_g A_h$  for all  $g$  and  $h$  in  $\mathcal{G}$ ,
- (ii)  $A_0 = 1$ , the identity operator on  $\mathcal{H}$ , and
- (iii) for each  $x$  in  $\mathcal{H}$ , the mapping  $g \rightarrow A_g x$  is continuous with respect to the norm topology on  $\mathcal{H}$ .

When it is reasonable to expect no great confusion to ensue, we will simply refer to the above as the strongly continuous group  $\{A_g\}$ . The group  $\{A_g\}$  is called a unitary group if  $A_g$  is a unitary operator for every  $g$  in  $\mathcal{G}$ . Similar inter-