

# On a Coerciveness Theorem by Schulenberger and Wilcox

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**1. Introduction.** Let  $X, Y$  be Banach spaces such that  $Y \subset X$ , with the inclusion continuous and dense. We denote by  $\|\cdot\|_X$  and  $\|\cdot\|_Y$  the associated norms.

Let  $A$  be a closed linear operator with domain  $D(A)$  and range  $R(A)$  both in  $X$ . Let  $N = N(A)$  be the null space of  $A$ , and let  $M$  be a (closed) subspace of  $X$  complementary to  $N$ . We say  $A$  is *coercive* on  $M$  (with respect to  $X, Y$ ) if

(C)  $u \in D(A) \cap M$  implies  $u \in Y$  and  $\|u\|_Y \leq c(\|Au\|_X + \|u\|_X)$ ,

where  $c$  is a constant. In this case we shall also say  $M$  is *admissible* (for  $X, Y, A$ ).

The (partial) coerciveness defined here has important applications in scattering theory (see Schulenberger and Wilcox [1]). In [2, 3] Schulenberger and Wilcox show that  $N^\perp$  is admissible if  $A$  is a first-order system of partial differential operators satisfying certain conditions, where  $X$  is a Hilbert space. Their results were subsequently generalized by Sarason [4] with a simplified proof.

The purpose of the present paper is to deduce some general conditions for a given subspace  $M$  to be admissible. In particular, we shall give a recipe for producing admissible subspaces  $M$  from a particular one  $M_0$ . It turns out that this is a geometric problem in a Banach space and has little to do with the detailed structure of the operator  $A$ . As an application of our results, we shall give another proof of the S-W-S theorem with some generalizations (including an  $L^p$ -theory, see Theorem 3.1 and Remark 3.2).

**2. Conditions for coerciveness.** It is convenient to rewrite condition (C) by introducing the projection  $P$  of  $X$  onto  $M$  along  $N$ . It is readily seen that (C) is equivalent to

(C')  $u \in D(A)$  implies  $Pu \in Y$  and  $\|Pu\|_Y \leq c(\|Au\|_X + \|u\|_X)$ .

**Theorem 2.1.** *In order that a subspace  $M$  of  $X$ , complementary to  $N$ , be*