

# Logarithms of $L^p$ Translations

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**1. Introduction.** A bounded linear operator  $T$  on a Banach space  $X$  is said to have a logarithm if there exists a bounded linear operator  $S$  on  $X$  such that  $\exp S = T$ . A necessary condition for an operator to have a logarithm is that it be invertible, whereas a simple sufficient condition for a logarithm to exist is that 0 be in the unbounded component of the resolvent set of the operator (see [4], Sec. 5.4). However, if  $T$  is invertible and 0 is in a bounded component of its resolvent set, no general deduction can be made about the existence of a logarithm of  $T$ .

Given a locally compact abelian group  $G$  and  $p$  in the range  $1 \leq p \leq \infty$ , the translation operator on  $L^p(G)$  associated with an element  $x$  in  $G$  is the bounded linear operator  $R_x$  on  $L^p(G)$  defined by

$$(R_x f)(y) = f(y + x),$$

where  $f \in L^p(G)$  and  $y \in G$  a.e. (locally a.e. in the case when  $p = \infty$ ). Each such translation operator is an invertible isometry. It is shown in [3] that, if  $x$  is an element of infinite order in the circle group  $\mathbf{T}$ , then the translation operator on  $L^1(\mathbf{T})$  associated with  $x$  does not have a logarithm. In contrast, the main result of this paper (Theorem 2) asserts that, for an arbitrary locally compact abelian group  $G$  and an arbitrary element  $x$  in  $G$ , the translation operator  $R_x$  on  $L^p(G)$  has a logarithm whenever  $1 < p < \infty$ . This is of principal interest when  $x$  has infinite order in  $G$ . If  $x$  has finite order, then the spectrum of  $R_x$  consists of a finite set of roots of unity and so does not separate the plane. Hence a logarithm exists by the above sufficiency condition. However, it can be shown (Theorem 1) that, if  $x$  has infinite order, then the spectrum of  $R_x$  equals the unit circle, and so this sufficiency condition cannot be applied. The proof of the existence of a logarithm in fact relies on several results about multipliers for  $L^p(G)$ , and these are discussed in §2. Finally, conditions are given on the spectrum of a logarithm of  $R_x$  under which it is unique.

**Notation.** Throughout,  $G$  is an arbitrary locally compact abelian group with dual group  $\Gamma$ . The value of  $\gamma \in \Gamma$  at  $x \in G$  is written as  $\langle x, \gamma \rangle$ . The integers,