

The Singular Cauchy Problem for Ordinary Differential Equations

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1. Introduction. Let $f: D \rightarrow \mathbf{R}^n$ be a vector-valued function defined on a domain (open connected set) $D \subset \mathbf{R}^{1+n}$ and taking values in \mathbf{R}^n , Euclidean n -space. The usual problem in ordinary differential equations consists of seeking an interval I and a function $x \in C[I, \mathbf{R}^n]$ which satisfies the equation

$$(1) \quad x'(t) = f(t, x(t))$$

for all $t \in I$, and passes through some preassigned point (t_0, x_0) lying in the interior of D . That is, $t_0 \in I$ and

$$(2) \quad x(t_0) = x_0.$$

The most elementary result regarding the above problem is the Cauchy–Peano existence theorem [4, p. 6] which states that continuity of f on D assures the existence of at least one solution to (1), (2). Such a solution also belongs to C^1 by the continuity of f .

However, if one only requires a solution to be an absolutely continuous function satisfying (1) for almost all $t \in I$, a theorem of Carathéodory [8, p. 28] provides existence under weaker conditions. It suffices if, on D , f is measurable in t for fixed x , continuous in x for fixed t , and for each compact $u \subset D$ there is an integrable function $m_u(t)$ with $\|f(t, x)\| \leq m_u(t)$ for all $(t, x) \in u$, for some norm $\|\cdot\|$.

In the present paper we consider the problem in the sense of Carathéodory, but the initial point is allowed to lie on the boundary of the set D . Such a problem arises in a two-body problem of electrodynamics [19].

The case where the initial point (t_0, x_0) lies on the boundary of D has been studied by a number of authors [3], [6], [7], [10], [11], [17] and has been referred to as the singular Cauchy problem [10], [11], or the limit Cauchy problem [17].

In the present paper a theorem generalizing the result of Carathéodory is presented, and its relation to known results on the singular Cauchy problem is discussed. We also prove that the hypotheses of a special case of our existence