

Localness of Certain Banach Modules

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1. Let X be a compact subset of the complex plane \mathbf{C} , and let $\mathcal{R}(X)$ denote the space of C^∞ functions on \mathbf{C} which agree on a neighbourhood of X with rational functions. For $0 \leq m \in \mathbf{Z}$, let $\overline{\mathcal{P}}_m$ denote the space of conjugate-analytic polynomials of degree at most m . Then for $\beta > 0$, $[\mathcal{R}]_{\beta, X}$ (the closure in $\text{Lip}(\beta, X)$ of $\mathcal{R}(X)$) is a Banach algebra, and $[\mathcal{R} \overline{\mathcal{P}}_m]_\beta$ is a finitely generated $[\mathcal{R}]_\beta$ -module. These modules arise naturally in the study of rational approximation in high Lipschitz norms [7]. The purpose of this note is to show that $[\mathcal{R} \overline{\mathcal{P}}_m]_\beta$ is *locally determined* in a certain sense, for $\beta \notin \mathbf{Z}$. Precisely speaking, the result is as follows:

Theorem. *Let $0 < \beta \notin \mathbf{Z}$, $0 \leq m \in \mathbf{Z}$, and let X be a compact subset of \mathbf{C} . Suppose $f : X \rightarrow \mathbf{C}$ and every point $a \in X$ has a closed neighbourhood U such that*

$$f \in [\mathcal{R}(X \cap U) \overline{\mathcal{P}}_m]_{\beta, X \cap U}.$$

Then

$$f \in [\mathcal{R}(X) \overline{\mathcal{P}}_m]_{\beta, X}.$$

The first result of this kind was Bishop's theorem [3, p. 51] that *the uniform closure on X of $\mathcal{R}(X)$ is local, for X compact in \mathbf{C}* . Kallin [4] showed that this fails for sets in higher dimensional \mathbf{C}^n . Weinstock [9] proved a localness theorem for approximation by solutions of elliptic partial differential equations. He worked in the space of germs of C^k functions, where k is less than the order of the operator. These germ spaces are unsuitable for approximation at orders at and beyond the order of the operator. The localness result of the present paper is the first which involves approximation beyond the order of the operator. We conjecture that Weinstock's results can be extended to all nonintegral orders if one formulates them in terms of $\text{Lip } \beta$ spaces instead of germ spaces.

The question whether $[\mathcal{R} \overline{\mathcal{P}}_m]_\beta$ is local for *integral* $\beta > 0$ remains open. There is some evidence in favor of localness: the necessary and sufficient condition for

$$[\mathcal{R}]_1 = [\mathcal{E}]_1$$

which is given in [5], is a local condition.