

The Capacitor Problem

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Let Ω be a bounded domain in \mathbf{R}^3 whose boundary Γ is C^∞ and connected. Suppose R is an open subregion of Ω which can be written in the form

$$(1) \quad R = \Omega \sim K,$$

where K is a compact subset of Ω . For such open sets we define

$$(2) \quad V_R(x) = \int_R \frac{dy}{|x - y|},$$

the electrostatic potential produced by a charge of unit density distributed over R . In this paper we consider the following free boundary value problem: Given a real number q satisfying $0 < q < m(\Omega)$, the volume of Ω , find a region R of the form (1) with $m(R) = q$ such that

$$(3) \quad V_R \text{ is constant on } \partial K.$$

Specifically, we prove that providing q is sufficiently small, there is a region R with $m(R) = q$ which satisfies (3) such that ∂K , the inner boundary of R , is a surface of class C^∞ which is diffeomorphic to Γ . Moreover we show that the inner boundary varies smoothly with q .

The problem was encountered by Lieb and Lebowitz [5] in their study of the thermodynamic limit in statistical mechanics of systems that have a net electrical charge. It may also be formulated variationally—the solution region R minimizes the electrostatic potential energy,

$$E = \int_R dx \int_R dy \frac{1}{|x - y|},$$

among all measurable subsets of Ω subject to the constraint that $m(R) = q$. Using the variational approach Wils [9] proved the existence and uniqueness of the region R , but was not able to obtain any information about its regularity. We base our work on (3) directly: we formulate (3) as a non-linear functional equation and we solve the resulting equation with an implicit function theorem (see for example [6]). We could also obtain a uniqueness theorem from our