

# *Semigroups of Linear Operators on Locally Convex Spaces with Banach Subspaces*

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**I. Introduction.** In [2] and [3], J. W. Neuberger pointed out that to a certain extent the theories of nonlinear semigroups and semidynamical systems can be subsumed into linear semigroup theory. To wit, suppose that  $S$  is a metric space,  $\mathbf{R}^+ = [0, \infty)$ , and  $\pi$  is a continuous function from  $\mathbf{R}^+ \times S$  to  $S$  such that  $\pi(s, \pi(t, p)) = \pi(s + t, p)$  and  $\pi(0, p) = p$  whenever  $(s, t, p)$  is in  $\mathbf{R}^+ \times \mathbf{R}^+ \times S$ . Let  $H$  be a Banach space, and let  $Q$  be the Banach space of all bounded continuous (or uniformly continuous) functions from  $S$  to  $H$ , with supremum norm. If  $t$  is in  $\mathbf{R}^+$ , let  $T(t)$  from  $Q$  to  $Q$  be given by  $T(t)[f](p) = f(\pi(t, p))$ . Now  $T$  is a semigroup of nonexpansive linear operators over  $Q$ . As Neuberger pointed out, however,  $T$  is in general not strongly continuous anywhere, and thus if one wishes to apply linear semigroup theory, one must first work out that theory for this circumstance.

If one replaces  $Q$  by the locally convex space of *all* continuous functions from  $S$  to  $H$ , topologized by uniform convergence on compact sets, then  $T$  is strongly continuous. But for the purpose of inserting this situation into the standard theory, this is no aid, since now  $T$  need not be equicontinuous (see Yosida [4, Chapter IX] for the equicontinuous theory in locally convex spaces). The purpose of the present work is to develop a linear theory which includes Neuberger's circumstance, and which is of interest in its own right.

**II. A semigroup theory.** Let  $X$  be a linear space over the real or complex field, let  $\Gamma$  be a set, let  $p$  be a function on  $\Gamma$  each value of which is a seminorm on  $X$ , and suppose that  $X$ , topologized by  $\{p_\gamma : \gamma \text{ is in } \Gamma\}$ , is a sequentially complete locally convex linear topological space. Let  $Y$  be the subset of  $X$  to which  $x$  belongs if and only if  $\sup\{p_\gamma(x) : \gamma \text{ is in } \Gamma\}$  is finite. If  $x$  is in  $Y$ , let  $|x|$  be this supremum. Now  $|\cdot|$  is a norm on  $Y$  and, since  $X$  is sequentially complete,  $Y$  is a Banach space with respect to  $|\cdot|$ . Let  $\mathfrak{L}[X]$  be the algebra of  $X$ -continuous linear functions from  $X$  to  $X$ , and let  $\mathfrak{B}[Y]$  be the algebra of