

# Fundamental Groups of Plane Curves and their Duals

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Let  $C$  be an algebraic curve in the complex projective plane. The study of the fundamental group  $\pi_1(\mathbf{P}^2 - C)$  was initiated by Enriques, Lefschetz, Zariski [3] and van Kampen [1], and remains an active common ground for topology and algebraic geometry. A well-known conjecture in this area says that if  $C$  is an irreducible plane curve of degree  $n$  whose only singularities are ordinary double points, then  $\pi_1(\mathbf{P}^2 - C)$  is cyclic of order  $n$ . Zariski showed that this would follow from results on deformations of singular curves announced by Severi, but the latter's proofs are considered incomplete. This note establishes a "dual" to the conjecture: the fundamental group of the curve *dual* to  $C$  is *not* cyclic.

More precisely, the lines in  $\mathbf{P}^2$  form a projective space  $(\mathbf{P}^2)^*$  which is isomorphic to  $\mathbf{P}^2$ . If  $C \subset \mathbf{P}^2$  is an algebraic curve, then the lines tangent to  $C$  form an algebraic curve  $C^*$  in  $(\mathbf{P}^2)^*$ . If  $C$  is irreducible then  $C^*$  is also, and conversely. (See Walker [2].) Our result is:

**Theorem.** *Let  $C \subset \mathbf{P}^2$  be an irreducible curve of degree  $n$  whose only singularities are ordinary double points. Then  $\pi_1((\mathbf{P}^2)^* - C^*)$  maps onto  $S_n$ , the symmetric group on  $n$  letters.*

For  $n > 2$  this shows immediately that  $\pi_1((\mathbf{P}^2)^* - C^*)$  cannot be abelian, or even nilpotent.

The map to the symmetric group is obtained by exhibiting  $C$  as a branched cover by means of a "Lefschetz pencil." Choose a line  $L^*$  in  $(\mathbf{P}^2)^*$  which represents the pencil of lines through some point  $p$  in  $\mathbf{P}^2 - C$ . Map  $C$  to  $L^*$  by sending each  $x$  in  $C$  to the line joining  $x$  to  $p$ . This branches  $C$  over  $L^*$  and the branch points are given by the "singular lines" in the pencil—those lines which are tangent to  $C$  or which pass through the singularities of  $C$ . We can identify the singular lines as points in  $(\mathbf{P}^2)^*$  as follows: Lines tangent to  $C$  are dual to points of  $C^*$ . Lines through the singularities of  $C$  are dual to points lying on multiple tangents to  $C^*$ . For example, lines through a double point of  $C$  are dual to points on a double tangent to  $C^*$ , and lines through a cusp of  $C$  are dual to points on a flex tangent to  $C^*$ .