

Gaussian Sums over $GL(2, \mathbf{Z}/N)$ and Hecke Operators II

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We continue to investigate Hecke operators of type R , where R denotes a primitive irreducible representation by complex matrices of $GL(2, \mathbf{Z}/N)$. We showed earlier [2] that, for certain representations, these Hecke operators satisfy identities of a simple nature, and exhibited a rational formula for their associated Dirichlet series. Shimura [3] had introduced this series in a more general setting, and proved, among other things, that it satisfies a functional equation. Our objective here is to employ the functional equation together with the identities in order to get additional information about the Hecke operators. In particular, we shall prove “periodic relations”, compute eigenvalues, and obtain a more natural rational formula for the Dirichlet series.

The Dirichlet series $\mathfrak{W}(s)$, defined in [3], has a meromorphic continuation to the whole complex plane, and satisfies the functional equation:

$$\mathfrak{W}(s)\mathfrak{W}(2-s) = \Lambda$$

where Λ is a certain scalar operator ([3], p. 275, formula (42)). Corresponding to the prime factorization $p_1^{n_1} \cdots p_r^{n_r}$ of N , there is a splitting of $\mathfrak{W}(s)$ into a product of Dirichlet series $\mathfrak{W}(s, p_1) \cdots \mathfrak{W}(s, p_r)$. We wish to specialize to the case in which N is a prime power p^n , because in that case $\mathfrak{W}(s, p)$ satisfies the functional equation:

$$\mathfrak{W}(s, p)\mathfrak{W}(2-s, p) = \Lambda.$$

Several other simplifications arise: referring to notions defined in [2], we see that the Gaussian sums $W(A)$, $W(A, p)$, and $W(A, p)'$ all coincide, that the Hecke operators $T(A)$ and $T(A, p)$ coincide, and that the Dirichlet series $\mathfrak{W}(s)$ and $\mathfrak{W}(s, p)$ coincide. Therefore, for maximum clarity and simpler notation, it seems desirable to briefly reestablish some notation and recall a few facts, taking into account that N is p^n .

1. Definitions and statement of the results. Henceforth, A shall denote a matrix in $M(2, \mathbf{Z})$ having positive determinant, Γ shall denote $SL(2, \mathbf{Z})$, g shall denote a matrix in $GL(2, \mathbf{Z}/p^n)$, and H shall denote the upper half of the complex