

# Weakly Convergent Sequences in $L_\infty$

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In [2], Tullio Zolezzi proved that in a positive measure space  $(X, \alpha, \mu)$ , with  $X \in \alpha$  and  $\mu(X) < \infty$ , a weakly convergent sequence in  $L_\infty$  is  $\|\cdot\|_p$ -convergent in  $L_p$  ( $1 \leq p < \infty$ ). In this note we show that this result is an immediate consequence of the fact that  $L_\infty$  satisfies the property D.P. strict ([1], p. 139). Also it easily follows from this that the result is also true for any  $p$ , with  $0 < p < \infty$ .

**Theorem.** *Let  $(X, \alpha, \mu)$ , with  $X \in \alpha$  and  $\mu(X) < \infty$ , be a positive measure space and a sequence  $f_n \rightarrow f$  weakly in  $L_\infty$ . Then  $f_n \rightarrow f$  in  $(L_p, \|\cdot\|_p)$  ( $0 < p < \infty$ ).*

*Proof.* Without loss of generality we take  $f = 0$ . Take  $p$ ,  $1 \leq p < \infty$ . The canonical mapping  $(L_\infty, \|\cdot\|_\infty) \rightarrow (L_p, \|\cdot\|_p)$  is easily seen to be weakly compact and since  $L_\infty$  satisfies D.P. strict,  $\|f_n\|_p \rightarrow 0$ . In particular  $\|f_n\|_1 \rightarrow 0$ . Take  $q$ ,  $0 < q < 1$ . If  $\|f_n\|_q \rightarrow 0$ , by taking subsequences, if necessary, we can assume that  $\|f_n\|_q \geq \eta > 0$ ,  $\forall n$ . Taking a subsequence if necessary and using the result that  $\|f_n\|_1 \rightarrow 0$ , we get  $f_n \rightarrow 0$  a.e. which implies that  $|f_n|^q \rightarrow 0$  a.e. Using the fact that  $\{f_n\}$  are uniformly bounded in  $\|\cdot\|_\infty$  (in a Banach space a weakly bounded set is norm-bounded), by dominated convergence theorem,  $\|f_n\|_q \rightarrow 0$ , a contradiction. This proves the result.

## REFERENCES

1. A. GROTHENDIECK, *Sur les applications lineaires faiblement compactes d'espaces du type  $C(K)$*  Canad. J. Math. **5** (1953), 129–173.
2. TULLIO ZOLEZZI, *On weak convergence in  $L_\infty$* , Indiana Univ. Math. J. **23** (1974), 765–766.

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