

A Remarkable Inequality and the Uniform Convergence of Fourier Series

ADRIANO M. GARSIA

Introduction. Let $f(x)$ be a measurable function on $[0, 1]$ and let $f^*(x)$ denote the non-increasing rearrangement of f . That is $f^*(x)$ is the unique right-continuous non-increasing function on $[0, 1]$ such that for all real λ

$$m\{x: f^*(x) \geq \lambda\} = m\{x: f(x) \geq \lambda\}.$$

If for some $p > 1$ f is in $L_p[0, 1]$ we set for $0 < \delta < 1$

$$\text{I.1} \quad Q_p(\delta, f) = \left[\frac{1}{\delta} \iint_{|x-y| \leq \delta} |f(x) - f(y)|^p dx dy \right]^{1/p}.$$

This, as we shall see, is the most natural way of writing the L_p modulus of continuity of a non-periodic function.

The inequality we shall present in this paper can be used to derive information as to the smoothness and further integrability properties of f from the behavior of $Q_p(\delta, f)$ as $\delta \rightarrow 0$.

It can be stated as follows:

Theorem I.1. *If $f \in L_p[0, 1]$ then for each $0 < x \leq 1/2$ we have*

$$\text{I.2} \quad \left. \begin{array}{l} f^*(x) - f^*(1/2) \\ f^*(1/2) - f^*(1-x) \end{array} \right\} \leq \frac{4^{1/p}}{\log 3/2} \int_x^1 Q_p(\delta, f) \frac{d\delta}{\delta^{1+1/p}}.$$

To understand the significance of this result we shall state some of its most interesting consequences.

Theorem I.2. *Let $f \in L_p[0, 1]$ and suppose that $Q_p(\delta, f)/\delta^{1+1/p}$ is integrable on $[0, 1]$, then f is essentially continuous and for almost all $x, y \in [0, 1]$ we have*

$$\text{I.3} \quad |f(x) - f(y)| \leq \frac{2^{1+2/p}}{\log 3/2} \int_0^{|x-y|} Q_p(\delta, f) \frac{d\delta}{\delta^{1+1/p}}.$$

Of course, by a change of scale argument, these inequalities can be stated for any given interval. In particular, if f is periodic of period 2π and the L_p