

Characterization of C^n -operators

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Introduction & statement of results. Let A be a Banach algebra with identity, and let $C^n(\mathbf{R})$ be the topological algebra of all C^n -functions on the real line \mathbf{R} . The element $a \in A$ is of class $C^n(\mathbf{R})$ if there exists a continuous homomorphism of $C^n(\mathbf{R})$ into A which sends $\phi(t) = t$ to a . Let

$$E_n(t, a) = \sum_k (it)^k a^{k+n}/(k+n)!$$

and $v_n(a) = \sup \|\sum_t \phi(t)E_n(t, a)\|$, where the sup is taken over all $\phi : \mathbf{R} \rightarrow C$ with finite support for which $\|\phi\|_\infty \leq 1$.

Theorem 1. a is of class $C^n(\mathbf{R})$ iff $v_n(a) < \infty$. Theorem 2 characterizes $C^n(\mathbf{R})$ -operators in Hilbert space with positive characteristic measure. Theorem 3 gives a localized version of Theorem 1.

It follows from the definition of the topology on $C^n(\mathbf{R})$ that $a \in A$ is of class $C^n(\mathbf{R})$ iff there exists a closed interval Δ and a continuous homomorphism τ_Δ of the Banach algebra $C^n(\Delta)$ into A , which sends $\phi(t) = t$ onto a ; given Δ , τ_Δ is unique, and is called the $C^n(\Delta)$ -operational calculus for a (cf. [6]). Any closed interval $\Delta^1 \supset \Delta$ has the same property as Δ , and therefore, given any bounded set $Q \subset \mathbf{R}$, we may assume without loss of generality that Δ is symmetric ($\Delta = -\Delta$) and contains Q . τ_Δ is related to the (unique) $C^n(\mathbf{R})$ -operational calculus τ by $\tau(f) = \tau_\Delta(f|_\Delta)$, and we may then use the same notation τ for both.

The known characterizations of elements of class $C^n(\mathbf{R})$ were obtained by selecting dense subsets of $C^n(\Delta)$ on which τ is explicitly known (cf. [6]). However the criteria thus obtained are "non-intrinsic" in as much as Δ keeps appearing. The "intrinsic" criteria presented here generalize the known result for $n = 0$ (cf. [5], Theorem 6). The role of the group e^{ita} is now taken by the function

$$E_n : \mathbf{R} \times A \rightarrow A$$

defined by

$$(1) \quad E_n(t, a) = \sum_{k=0}^{\infty} (it)^k a^{k+n}/(k+n)!.$$

Clearly E_n is continuous, and $E_0(t, a) = e^{ita}$.