

Spaces of Harmonic Functions Representable by Poisson Integrals of Functions in BMO and $\mathcal{L}_{p,\lambda}$

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Introduction. Several function spaces arise as the space of traces of solutions of a linear partial differential equation. This feature has been fully studied for “global constraints” in references [6], [10] and [11], and for “local constraints”, in the sense that at least some type of pointwise limit exists almost everywhere, in [1] and [10].

In the course of proving the fundamental step in the duality of H^1 and BMO , Fefferman showed that a function f of bounded mean oscillation is the trace of the solution of $\Delta u + u_{tt} = 0$, $u(x, 0) = f(x)$, where u satisfies

$$(0.1) \quad \sup_{h, x_0} h^{-n} \int_0^h \int_{|x-x_0| \leq h} t |\nabla u(x, t)|^2 dt dx \leq C.$$

Expanding on this result, we prove that, conversely, condition (0.1) characterizes all the harmonic functions whose traces are in BMO . Moreover, since BMO is contained in the $\mathcal{L}_{p,\lambda}$ scale of function spaces (see [7]), we study the characterization of $\mathcal{L}_{p,\lambda}$ by such a condition, and derive as a consequence a new easy proof of the Campanato–Meyers theorem. In a second paper, we will continue with the characterization problem for $\mathcal{L}_{p,\lambda}$ and consider the corresponding questions for solutions of the heat equation. Our notation is standard, following closely Stein’s book [10], and Fefferman–Stein [4].

§I. The space HMO and its characterization. Given any x_0 in \mathbf{R}^n and $\delta > 0$, we denote by Q_δ the cube (or ball) with center at x_0 and side (resp. radius) of length δ , and we recall that a locally integrable function f is in $BMO = BMO(\mathbf{R}^n)$ if

$$\|f\|_* = \sup \left\{ \frac{1}{|Q_\delta|} \int_{Q_\delta} |f(x) - f_{Q_\delta}| dx : x_0 \in \mathbf{R}^n, \delta > 0 \right\} < \infty$$

where $|Q_\delta|$ is the Lebesgue measure of Q_δ , and f_{Q_δ} denotes the average (or mean value) of f on Q_δ . It is well-known that, identifying f with $f + C$ (for any constant C), BMO is a Banach space with norm $\|\cdot\|_*$.