

The Free Boundary Determined by the Solution to a Differential Equation

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Introduction. The study of variational inequalities has led to free boundary value problems for solutions of elliptic equations where one desires to conclude the smoothness of the free boundary and some additional properties. The conclusion follows from relations satisfied by some function on the free boundary.

As a prototype of this question, assume given a bounded simply connected domain ω in the $z = x_1 + ix_2$ plane whose boundary $\partial\omega$ contains a Jordan arc γ . Suppose that $u \in C^1(\omega \cup \gamma)$ is harmonic in ω , ψ is $C^{2,\lambda}$ in a neighborhood of $\bar{\omega}$, and that

$$u_{x_j} = \psi_{x_j} \quad \text{for } z \in \gamma, \quad j = 1, 2,$$

where

$$|\Delta\psi| \geq \alpha_0 > 0 \quad \text{in } \bar{\omega}.$$

We now wish to conclude that γ is a smooth curve in the sense that a conformal mapping φ onto ω from a semidisc $G = \{t : |t| < 1, \text{Im } t > 0\}$ with the property that φ maps $-1 < t < 1$ onto γ has a Hölder continuous derivative on $-1 < t < 1$. In what follows, we shall rephrase the question in terms of a complex valued function $g(z)$ which vanishes on γ and satisfies $g_{\bar{z}} = \alpha$ in ω , $|\alpha(z)| \geq \alpha_0 > 0$. The function g corresponds, obviously, to $(1/2)(u - \psi)_{x_1} - (i/2)(u - \psi)_{x_2}$.

This work differs from [7]. Most importantly, the integrability of φ' above, previously determined in an *ad hoc* manner in individual cases, arises as a general characteristic (Lemma 1.1). A refined smoothness theorem which provides an asymptotic expansion for the conformal mapping is proved in §4. It depends on the nature of certain functions discussed in §3.

The major theorems of this paper have been applied in [9] where an overdetermined boundary value problem is formulated and solved. This problem motivated our study. The conclusions here may also be applied to other problems, for example to prove Theorem 2 of [7] or to discuss the free surface in certain filtration problems (cf. C. Baiocchi [1] or V. Benci [2]).

Similar questions concerned with the case of analytic data have been discussed in H. Lewy and G. Stampacchia [11] and by H. Lewy [10].