

# The Representation of Monotonous Processes by Exponentials

WALTER NOLL

**1. Statement of results.** Let  $\mathfrak{U}$  be a finite-dimensional real vector space. The space  $\text{Lin}(\mathfrak{U}, \mathfrak{U}^*)$  of all linear mappings from  $\mathfrak{U}$  into its dual  $\mathfrak{U}^*$  can be identified with the space of all bilinear forms on  $\mathfrak{U}$ . The subspace of  $\text{Lin}(\mathfrak{U}, \mathfrak{U}^*)$  corresponding to the symmetric bilinear forms on  $\mathfrak{U}$  is denoted by  $\text{Sym}(\mathfrak{U}, \mathfrak{U}^*)$ , and the set of positive definite elements of  $\text{Sym}(\mathfrak{U}, \mathfrak{U}^*)$  by  $\text{Sym}^+(\mathfrak{U}, \mathfrak{U}^*)$ . By a *deformation process* for  $\mathfrak{U}$  we mean a function

$$P : [0, d] \rightarrow \text{Sym}^+(\mathfrak{U}, \mathfrak{U}^*)$$

from some closed interval  $[0, d]$ ,  $d \geq 0$ , into  $\text{Sym}^+(\mathfrak{U}, \mathfrak{U}^*)$ . The number  $d$  is called the *duration* of the process. If  $[t_1, t_2] \subset [0, d]$ , we define the *segment*

$$P_{[t_1, t_2]} : [0, t_2 - t_1] \rightarrow \text{Sym}^+(\mathfrak{U}, \mathfrak{U}^*)$$

of the process  $P$  by

$$P_{[t_1, t_2]}(t) = P(t + t_1) \quad \text{when } t \in [0, t_2 - t_1].$$

It is a process of duration  $t_2 - t_1$ .

We say that the processes  $P_1$  and  $P_2$  are *congruent* if they have the same duration  $d$  and if there is an element  $A$  in the group  $\text{Invl}(\mathfrak{U})$  of all invertible linear transformations of  $\mathfrak{U}$  such that

$$P_1(t) = A^*P_2(t)A \quad \text{for all } t \in [0, d],$$

where  $A^* \in \text{Invl}(\mathfrak{U}^*)$  denotes the adjoint of  $A$ . We say that a process  $P$  is *monotonous* if it is continuous and if any two segments of  $P$  of equal duration are congruent.

The main purpose of this paper is to prove the following representation theorem:

**Theorem 1.** *A process  $P$  for  $\mathfrak{U}$ , of duration  $d$ , is monotonous if and only if there exists an  $E$  in the algebra  $\text{Lin}(\mathfrak{U})$  of linear transformations of  $\mathfrak{U}$  such that*

$$(1) \quad P(t) = \exp(tE^*)P(0)\exp(tE) \quad \text{for all } t \in [0, d].$$