

# $L^p$ -Isometries and Equimeasurability

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## 1. Introduction.

**1.1.** Throughout this paper,  $\mu$  and  $\nu$  will be finite positive measures (on sets  $X$  and  $Y$  that will play essentially no role at all). For  $0 < p \leq \infty$ ,  $L^p(\mu)$  and  $L^p(\nu)$  will be the usual Lebesgue spaces of complex functions. We shall be concerned with linear maps  $A$  that carry some subspace  $M$  of  $L^p(\mu)$  into  $L^p(\nu)$ , such that  $A1 = 1$  and

$$(1) \quad \int_Y |Af|^p d\nu = \int_X |f|^p d\mu$$

for all  $f \in M$ . (Thus, by assumption,  $1 \in M$ .)

The main results proved here show that any such  $A$  has a very strong equimeasurability property, at least when  $p$  is not an even integer. When  $M$  is all of  $L^p(\mu)$  (and  $p \neq 2$ ) this is not a new phenomenon [5]. However, equimeasurability holds when  $M$  is *any* subspace of  $L^p(\mu)$  which contains the constants, even a finite-dimensional one. This follows from Theorem I.

In Theorem II,  $M$  is assumed to be an algebra (with pointwise multiplication) and the principal additional conclusion is that  $A$  is then multiplicative.

The relations between Theorems I and II on the one hand, and earlier work of Frank Forelli [3], [4] and Robert Schneider [10] on the other, are discussed in Section 1.6. In Part 4, Theorem II is applied to the study of the isometries of  $H^p$ -spaces associated with polydisks  $U^n$  and with the euclidean unit ball  $B_n$  of  $\mathbb{C}^n$ , the space of  $n$  complex variables. The most interesting result obtained there (Theorem 4.2) is the complete description of the isometries of  $H^p(B_n)$  onto  $H^p(B_n)$ , for any  $p \neq 2$ , something which was previously known only for  $p > 2$  [4]. It was the search for this theorem about  $H^p(B_n)$  that led me to Theorems I and II. Some familiarity with  $B_n$  also turned out to be helpful in proving Lemma 2.5 for  $n > 1$ .

**1.2. Definition.** Suppose  $f_1, \dots, f_n$  are  $\mu$ -measurable functions on  $X$ ,  $g_1, \dots, g_n$  are  $\nu$ -measurable functions on  $Y$ , all with values in the complex field  $\mathbb{C}$ . Put  $F = (f_1, \dots, f_n)$ ,  $G = (g_1, \dots, g_n)$ . Regard the  $n$ -tuples  $F$  and  $G$  as  $\mathbb{C}^n$ -valued functions.