

# *Equivariant Embeddings of Riemannian Homogeneous Spaces*

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**1. Introduction.** A well-known theorem of Nash [7], [4] asserts that any Riemannian manifold possesses an *isometric* embedding in some Euclidean space. On the other hand, a basic theorem of Mostow and Palais [2, p. 315] states that if  $M$  is a compact manifold on which a compact Lie group acts, then  $M$  possesses an *equivariant* embedding in some Euclidean space. It seems natural to pose the question: If  $M$  is a compact Riemannian manifold,  $G$  a compact Lie group which acts on  $M$  by isometries, does there exist an embedding from  $M$  into some Euclidean space, which is both equivariant *and* isometric? The main theorem of this article says that the answer is “yes” when  $M$  is a compact Riemannian homogeneous space,  $G$  its entire group of isometries:

**Theorem.** *Any compact Riemannian homogeneous space possesses an embedding in some Euclidean space which is both equivariant and isometric.*

It has been known for some time that any compact irreducible Riemannian *symmetric* space possesses an equivariant isometric embedding in some Euclidean space (Cartan [3, §10]). In fact, Kobayashi [5] has constructed simple explicit equivariant isometric embeddings of many of the symmetric spaces.

The theorem is not true without the compactness assumption. Indeed, Robert Greene called our attention to a fact proven in Bieberbach [1, §IV]: If  $M$  is the hyperbolic plane,  $G$  its group of isometries,  $M$  possesses no equivariant isometric embedding into any Euclidean space. Our thanks go to Robert Greene for explaining this counterexample to us, as well as for helpful correspondence on related topics.

One final remark. Before we proved the above theorem, we investigated the special case of left-invariant metrics on the group  $SO(3)$  of rotations of  $E^3$ . We found that given any *fixed* integer  $n$ , there exist left-invariant metrics on  $SO(3)$  which are not induced by any equivariant embedding from  $SO(3)$  into  $E^n$ . Although any left-invariant metric is induced by an equivariant embedding into *some* Euclidean space, the minimal dimension needed for the receiving Euclidean space may be astronomically large.