

# *The Initial Value Problem for Surface Waves Under Gravity, I: The Simplest Case*

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**1. Introduction.** There are very few occasions in which the equations of surface waves under gravity (“water waves”) are known to have a solution. Indeed, except for solutions obtained by the so-called “inverse method” [1], where one solves the differential equations and then finds an appropriate boundary to match, there are, I believe, only five known solutions: progressing waves in water of infinite depth [2] and finite constant depth [3], the solitary wave [4], and steady waves over periodic and over monotone bottoms [5].

In all these cases, the solution is either steady or can be made so by a change of coordinates. Again barring “inverse” solutions, there are, to my knowledge, *no* known unsteady, free surface flows under gravity. It is my purpose to remedy this by studying the most natural unsteady situation, the initial value problem.

The problem is nonlinear, of course, and even its linear version [6] is not completely solved (*cf.* the remarks in [7]), but existence of weak solutions of the linear problem has been proved in some generality [8]. (See also [7, 9, 10].)

As for the full nonlinear problem we solve here, what is usually considered to be even worse than its nonlinearity is the fact that it involves an unknown free boundary that is to be determined as part of the solution. However, this difficulty can be circumvented by using the so-called Lagrange representation of the flow, following the course of each fluid particle in time. That is what we use here, solving the Lagrange equations by expanding the unknown functions in powers of the time, computing successive coefficients, estimating these, and then showing that the resulting series converge. As we shall see, this method works quite well, although it has the disadvantage of requiring that the initial data be analytic. I know of no way to remove the restriction to analytic data.

Analyticity questions aside, however, the method used here is quite general, and seems to apply to unsteady flows over general (analytic) bottoms in any number of dimensions. Nevertheless, we restrict ourselves here to the very simplest case of two dimensional flow where the water initially fills a horizontal slab, since then the ideas stand out most clearly and simply. My desire not to clutter up the argument may be forgivable in view of Gilbarg’s description [11] of the problem of unsteady flows under gravity as “[*p*]robably the most difficult