

Infinite Dimensional Versions of a Theorem of Brickman-Fillmore

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L. Brickman and P. A. Fillmore proved in [1] that if S and T are commuting operators on a finite dimensional vector space such that every subspace invariant for S is also invariant for T , then $T = p(S)$ for some polynomial p . In this note we extend their result to (bounded) operators on Hilbert space and as was suggested in [4], the statement of our results involves the paraclosed invariant subspaces of operators and not just those which are closed. (Recall that a linear submanifold is said to be paraclosed if it is the range of an operator acting between two Hilbert spaces.)

We begin with a result involving contractions. For the theory of contractions on Hilbert space, we refer the reader to [9]. We let H^2 and H^∞ denote the usual Hardy spaces of holomorphic functions on the unit disk (cf. [9]).

Theorem 1. *Let S be a non- C_0 , completely non-unitary contraction on \mathfrak{H} and let T be an operator on \mathfrak{H} . Then $T = u(S)$ for some u in H^∞ if and only if T leaves invariant the range of any operator A which intertwines S with a completely non-unitary contraction (that is, $SA = AS_A$, where S_A is some completely non-unitary contraction).*

Proof. By [11] there exists a non-zero f_0 in \mathfrak{H} such that the restriction of S to the subspace

$$\mathfrak{H}_{f_0} = \bigvee_{n \in \mathbb{Z}_+} S^n f_0$$

is not of class C_0 . Such a vector will be called a (non C_0)-vector in this proof. Then by virtue of [10], there exists an injective operator A_{f_0} from H^2 to \mathfrak{H} such that

$$(1) \quad SA_{f_0} = A_{f_0}M_+, \quad \overline{A_{f_0}H^2} = \mathfrak{H}_{f_0},$$

where M_+ denotes the multiplication operator defined by $(M_+\varphi)(\lambda) = \lambda\varphi(\lambda)$ ($|\lambda| < 1$) for φ in H^2 . Then $A_{f_0}H^2$ is invariant for T and thus, by the closed graph theorem,

$$A_{f_0}^{-1}TA_{f_0} = N$$