

On the Solvability of Nonlinear Elliptic Boundary Value Problems

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Introduction. In this note we are concerned with the solvability of the mildly nonlinear elliptic boundary value problem

$$(0.1) \quad \begin{cases} (\mathcal{A}u)(x) + p(x, u(x)) = f(x), & x \in \Omega, \\ (\mathcal{B}u)(x) = q(x, u(x)), & x \in \Gamma. \end{cases}$$

Here Ω denotes a bounded domain in \mathbf{R}^N ($N \geq 1$) with smooth boundary Γ , and \mathcal{A} is a linear, uniformly elliptic second order differential expression. The linear boundary operator \mathcal{B} takes either the form $\mathcal{B}u = u/\Gamma$ (Dirichlet boundary condition), in which case the function q is supposed to be independent of u , or the form $\mathcal{B}u = \partial u/\partial n_a$ (boundary condition of Neumann type). Assuming the knowledge of a *lower solution* ϕ :

$$(0.2) \quad \begin{cases} \mathcal{A}\phi + p(\cdot, \phi) \leq f & \text{in } \Omega, \\ \mathcal{B}\phi \leq q(\cdot, \phi|_{\Gamma}) & \text{on } \Gamma, \end{cases}$$

and an *upper solution* ψ (reverse inequalities in (0.2)) with $\phi \leq \psi$ in Ω , and imposing sufficient regularity on the functions involved, the existence of a classical solution u to problem (0.1) with $\phi \leq u \leq \psi$ has been proved employing a monotone iteration scheme (cf. Amann [1] and the earlier work by Cohen, Keller and their associates cited therein). This result has shown to be very fruitful in the discussion of various problems arising in the theory of nonlinear diffusion processes and in nuclear or chemical reactor theory.

We present here a new approach to problem (0.1) which demands no regularity hypotheses. Instead of employing the maximum principle and searching for a classical solution, we apply the penalty method and follow the spirit of the modern theory of linear elliptic boundary value problems (Hilbert space approach). Imposing additional regularity on the given functions, we then obtain further smoothness of the solution by the known regularity theory for linear elliptic boundary value problems.

In Section 1 we treat the case of Dirichlet boundary conditions in detail. The corresponding result for Neumann or "third kind" boundary conditions is stated in Section 2.