

Weak Steepest Descent for Linear Boundary Value Problems

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1. Introduction. In a recent paper Nashed [4] used the method of steepest descent to obtain least squares solutions to a singular linear operator equation in a Hilbert space, thereby generalizing earlier work of Kantorovich [1] for positive definite bounded linear operators. He also described briefly the method of weak steepest descent, obtaining convergence by results of Petryshyn [5]. This method has been used to solve matrix equations by Tewarson [6] and Whitney and Meany [7].

The purpose of this paper is to adapt the method of weak steepest descent to provide least squares solutions to linear boundary value problems. The development illustrates the functional analysis structure of a differential operator which can be used to solve differential equations. We have chosen to describe this particular method because it permits a very simple and completely self-contained presentation of the theory. However, it should be pointed out that all of the results have analogues for the method of steepest descent.

Let (f, g) and $\|f\|$ denote the standard inner product and norm in the real Hilbert space $S = L^2[a, b]$, and let convergence in S be indicated by $f_i \rightarrow f$. For any operator L in S we denote its domain, range, and null space by $\mathfrak{D}(L)$, $\mathfrak{R}(L)$, and $\mathfrak{N}(L)$, respectively.

Let $\tau = \sum_{i=0}^n a_i(t)(d/dt)^i$ be an n th order formal differential operator with coefficients $a_i(t) \in C^\infty[a, b]$ and with $a_n(t) \neq 0$ on $[a, b]$, let B_1, \dots, B_k be a set of k linearly independent boundary values, and let L be the differential operator in S defined by

$$\mathfrak{D}(L) = \{f \in H^n[a, b] | B_i(f) = 0, i = 1, \dots, k\}, Lf = \tau f.$$

For a fixed function $g_0 \in S$ we consider the boundary value problem

$$(1) \quad Lf = g_0.$$

In §2 we introduce the Hilbert space adjoint L_* which arises naturally when $\mathfrak{D}(L)$ is made into a Hilbert space under an appropriate inner product. We prove that $L_* = L^*(LL^* + I)^{-1}$, where L^* is the differential operator adjoint