

Total Positivity of the Spline Collocation Matrix

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0. Definitions. For given positive integers n and k , and a given real non-decreasing sequence $\mathbf{t} := (t_i)_1^{n+k}$ with

$$t_i < t_{i+k}, \quad \text{all } i,$$

denote by $\mathbf{S}_{k,\mathbf{t}}$ the linear span of the n normalized B -splines $N_{1,k}, \dots, N_{n,k}$, given by the rule that, for each t ,

$$N_{i,k}(t) := ([t_{i+1}, \dots, t_{i+k}] - [t_i, \dots, t_{i+k-1}])(\cdot - t)_+^{k-1}.$$

Here,

$$[\rho_0, \dots, \rho_r]f$$

denotes the r th divided difference of the function f at the points ρ_0, \dots, ρ_r . The elements of $\mathbf{S}_{k,\mathbf{t}}$ are called *polynomial splines of order k with knot sequence \mathbf{t}* .

Let $\boldsymbol{\tau} := (\tau_i)_1^n$ be a strictly increasing real sequence. As is essentially shown by Schoenberg and Whitney [7], *there exists, for given f , exactly one $s \in \mathbf{S}_{k,\mathbf{t}}$ such that*

$$s(\tau_i) = f(\tau_i), \quad i = 1, \dots, n,$$

if and only if

$$N_{i,k}(\tau_i) \neq 0, \quad i = 1, \dots, n,$$

i.e., if

$$t_i < \tau_i < t_{i+k}, \quad i = 1, \dots, n.$$

Their proof is an appeal to much more general results in the same paper concerning the positivity of translation determinants. This theorem has been generalized by Karlin and Ziegler [3] to include the case of repeated or osculatory collocation, as stated in Theorem 1 below. The proof given here relies only on Rolle's Theorem and on the facts that

$$(1) \quad (d/dt) \sum_i \alpha_i N_{i,k} = (k-1) \sum_i \frac{\alpha_i - \alpha_{i-1}}{t_{i+k-1} - t_i} N_{i,k-1};$$