

# The $L^2$ Behavior of Eigenfunction Expansions

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**1. Introduction.** Let  $\tau$  denote the differential expression defined, for suitable functions  $u$ , by

$$(1.1) \quad \tau u = u^{(2)} + p_1(x)u^{(1)} + p_0(x)u,$$

where  $p_1$  and  $p_0$  are, at least, of class  $L^1(0, 1)$ . Let  $M$  and  $N$  denote two  $2 \times 2$  matrices of complex constants with at least two linearly independent columns between them, and let  $\hat{u}(x)$  denote the column vector whose components are  $u(x)$  and  $u^{(1)}(x)$ . Let

$$(1.2) \quad Uu = M\hat{u}(0) + N\hat{u}(1).$$

Let  $\Delta$  denote the subspace of  $L^2(0, 1)$  whose members  $u$  have absolutely continuous derivatives  $u^{(1)}$ , and such that  $\tau u \in L^2(0, 1)$ ,  $Uu = 0$ . We assume  $\Delta$  is dense in  $L^2(0, 1)$ . This is certainly true if  $p_0$  and  $p_1$  are bounded. Let  $L$  denote the linear operator defined on  $L^2(0, 1)$  by

$$(1.3) \quad Lu = \tau u, \quad u \in \Delta.$$

Assuming that the spectrum of  $L$  is countably infinite, we are concerned with the question of the convergence, in the norm of  $L^2(0, 1)$ , of the expansion of an arbitrary function  $f \in L^2(0, 1)$  in the series of eigenfunctions of  $L$ . We obtain the following result. Let  $G(x, t, \rho)$  denote the Green's function of  $L$ , where  $\rho$  is the complex parameter. By examining the growth of  $G(x, t, \rho)$  as  $|\rho| \rightarrow \infty$ , we obtain an integer  $\nu \geq 0$ . If  $\nu \geq 1$ , let  $H_\nu$  denote the collection of those functions  $f \in L^2(0, 1)$  such that  $f^{(\nu)}$  is in  $L^2(0, 1)$  and  $f^{(k)}(0) = f^{(k)}(1) = 0$  for  $0 \leq k \leq \nu - 1$ . Let  $H_0 = L^2(0, 1)$ . Assume that for  $|\rho|$  sufficiently large, the equation  $\tau u = -\rho^2 u$  has two linearly independent solutions  $u_1(x, \rho)$ ,  $u_2(x, \rho)$  such that

$$(1.4) \quad u_1(x, \rho) = e^{i\rho x}[\alpha_0(x) + \rho^{-1}\alpha_1(x) + \cdots + \rho^{-\nu}\alpha_\nu(x) + O(\rho^{-\nu-1})]$$

and similarly for  $u_2$ , where  $\alpha_j(x)$  has  $\nu - j$  derivatives. Then each function  $f$  in  $H_\nu$  can be expressed as the limit in the mean of its expansion in eigenfunctions of  $L$ .

If the coefficients  $p_0$ ,  $p_1$  are of class  $C^\infty[0, 1]$ , and if the boundary condition (1.2) is *Birkhoff regular* [3; p. 56], then  $L$  is known to be a *spectral operator* [2; Chapter 19]. (The result in [2] is for differential expressions of arbitrary order.) Consequently each function  $f$  in  $L^2(0, 1)$  has an  $L^2$  convergent eigen-