

Ascent, Descent, Nullity and Defect of Products of Operators

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1. Introduction. Let X be a Banach space and A and B be bounded linear operators on X . In [2, Problem 60 and 61] and [5, p. 264, Ex. 6] Halmos and Taylor, respectively, indicate that the nonzero points of the spectrum of AB and BA are equal, *i.e.*, $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$. See also [1] for corresponding results concerning unbounded operators.

The purpose of this paper is twofold. First to prove that for $\lambda \neq 0$ and A and B bounded linear operators the ascent, descent, nullity and defect of $BA - \lambda I$ is equal to that of $AB - \lambda I$ and to give some representative sufficient conditions for the ascent, nullity and defect to be the same when A and B are closed linear operators. Secondly to show, when A and B are bounded, that for various definitions of the essential spectrum, the nonzero points of the essential spectra of AB and BA coincide and to list some representative sufficient conditions which will insure that these results will remain true when it is only known that A and B are closed. Additionally, using results of Lay [4], it is shown that AB is a Riesz (Meromorphic) operator if and only if BA is a Riesz (Meromorphic) operator.

2. Main result. For T a closed operator on X , denote by $N(T)$, the null space of T ; $R(T)$, the range of T ; $\alpha(T)$, the ascent of T ; $\delta(T)$, the descent of T ; $n(T)$, the nullity of T ; $d(T)$, the defect of T . The closure of a set S will be denoted \bar{S} .

Theorem 1. *Let A and B be bounded operators on a Banach space X . For $\lambda \neq 0$*

- (i) $n(AB - \lambda I) = n(BA - \lambda I)$;
- (ii) $\alpha(AB - \lambda I) = \alpha(BA - \lambda I)$;
- (iii) $R(AB - \lambda I)$ is closed iff $R(BA - \lambda I)$ is closed;
- (iv) $\delta(AB - \lambda I) = \delta(BA - \lambda I)$;
- (v) $d(AB - \lambda I) = d(BA - \lambda I)$.

Proof. (i) If x is an eigenvector of $BA - \lambda I$, then $(BA - \lambda I)x = 0 = A(BA - \lambda I)x = (AB - \lambda I)Ax$. Since $\lambda \neq 0$, implies $Ax \neq 0$, Ax is an eigenvector of $(AB - \lambda I)$. Let $\{x_1, x_2, \dots, x_n\}$ be a set of linearly independent