

# *$L_p$ Saturation and Inverse Theorems for Modified Bernstein Polynomials*

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The Bernstein polynomials are not bounded operators on  $L_p$ . A modification by Kantarovič [5] given by

$$(1.1) \quad B_n^*(f, t) = \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} (n+1) \int_{k/n+1}^{k+1/n+1} f(u) du$$

is a bounded operator on  $L_p[0, 1]$ , and is also an approximation process, that is  $\|B_n^*(f, \cdot) - f(\cdot)\|_p = o(1)$  for  $f \in L_p$ .

We first investigate the rate of convergence of

$$\|B_n^* f - f\|_{L_p[a_1, b_1]}, \quad 0 < a < a_1 < b_1 < b < 1,$$

where  $B_n^* f \equiv B_n^*(f, t)$ , in terms of the modulus of continuity  $w_2(f, h, p, [a, b])$ , that is

$$(1.2) \quad w_2(f, h, p, [a, b]) \equiv \sup_{\tau \leq h} \|f(\cdot + \tau) - 2f(\cdot) + f(\cdot - \tau)\|_{L_p[a, b]}.$$

Bojanic and Shisha [3] recently related  $\|B_n^* f - f\|_{L_1}$  to

$$w(h) = \sup_{0 < \tau < h} \|f(x) - f(x + \tau)\|_{L_1[0, 1-\tau]}.$$

As ours is a result on  $L_p$ ,  $p \geq 1$  and deals with  $w_2(h)$  rather than  $w(h)$ , it is of a more general nature. However the estimate of [3] is global and the general question is still open in the global case. For the Bernstein polynomial

$$(1.3) \quad B_n(f, x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

T. Popoviciu and G. Lorentz [7, pp. 20–21] proved the analogous result for the maximum norm on  $C[0, 1]$ .

A saturation theorem is achieved in section 5, that is a theorem showing that

$$\|B_n^* f - f\|_{L_p[a, b]} = O\left(\frac{1}{n}\right)$$

is the best rate of convergence when  $x(1-x)f'(x) \neq C$  in  $[a, b]$  and determining the class of functions for which this rate is attained. Saturation results were