

Eigendistribution Expansions on Heisenberg Groups

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§1. Introduction. Let N be a connected, simply connected nilpotent Lie group. N acts on the space $C_c^\infty(N)$ of compactly supported C^∞ functions by $f^g(x) = f(g^{-1}xg)$ for $x, g \in N$. A distribution Θ on N is called invariant if $\Theta(f^g) = \Theta(f)$ for all $f \in C_c^\infty(N)$ and $g \in N$. For any irreducible unitary representation π of N , the operator

$$\pi(f) = \int_N \pi(x)f(x) dx, \quad f \in C_c^\infty(N)$$

is of trace class, and the map $f \rightarrow \text{trace } \pi(f)$ is an invariant distribution Θ_π called the *global character* of π [1]. Furthermore, Θ_π is tempered; *i.e.* Θ_π extends to the Schwartz space $\mathcal{S}(N)$, defined by the exponential map identification of N with Euclidean space. Hence there is a natural class of invariant tempered distributions on N .

In this paper we consider the converse question: can an arbitrary tempered invariant distribution on N be expanded in terms of characters of irreducible unitary representations? We give an affirmative answer in the case where N is locally isomorphic to a product of Heisenberg groups.

To illustrate the type of result obtained, let N be the 3-dimensional Heisenberg group with Lie algebra \mathfrak{n} spanned by x, y, t , such that $[x, y] = t$. Let x^*, y^*, t^* be a corresponding basis for \mathfrak{n}^* , the real dual space of \mathfrak{n} , so that (x, y, t) is the linear coordinate on \mathfrak{n}^* . For $f \in \mathcal{S}(N)$ let \tilde{f} denote Euclidean Fourier transform of f . Then any infinite dimensional irreducible unitary representation of N has global character Θ_s given by

$$\Theta_s(f) = \frac{1}{2} |s|^{-1} \iint \tilde{f}(x, y, s) dx dy$$

for some $s \in \mathbf{R}$, $s \neq 0$. (See §3 for more details.) Any finite dimensional irreducible unitary representation is one dimensional with global character of the form

$$\Theta_{x,y}(f) = \tilde{f}(x, y, 0)$$

for some $(x, y) \in \mathbf{R}^2$. Then for any $f \in \mathcal{S}(N)$, $|s|\Theta_s(f) \in \mathcal{S}(\mathbf{R})$ and $\Theta_{x,y}(f) \in \mathcal{S}(\mathbf{R}^2)$.