

# A Quasi-Anosov Flow that Is Not Anosov

CLARK ROBINSON

M. Hirsch asked if a diffeomorphism  $f$  has a compact submanifold  $V \subset M$  as a hyperbolic subset of  $M$ , does it follow that  $f$  restricted to  $V$  is Anosov [1]. Mañé Ramirez, [4], R. Sacker and G. Sell, [5] and [6], and J. Selgrade, [7], have all proved that if the periodic points of a flow or diffeomorphism  $f$  are dense in  $V$  and if  $f$  has a property called quasi-Anosov, then  $f$  is Anosov. See also [2, Corollary 5.2]. The question arises if a quasi-Anosov diffeomorphism is always Anosov even without the assumption on the periodic points. In this note, we give an example of a quasi-Anosov flow that is not Anosov. By an embedding result of Mañé, this gives a counter-example to the question of Hirsch. We do not know if our flow can arise as the restriction of an Anosov flow. Conversations with C. Conley were most helpful in the preparation of this paper.

**§1. Definitions.** A diffeomorphism  $f$  is *quasi-Anosov* if the fact that  $|Tf^n v|$  is bounded for all  $n \in \mathbf{Z}$  implies that  $v = 0$ . For a flow  $f^t$  on  $V$  with no zero, let  $N$  be a subbundle of  $TV$  such that  $N_x$  and

$$X(x) = \frac{d}{dt} f^t(x)|_{t=0}$$

span  $T_x V$ . Let  $\pi : TV \rightarrow N$  be the projection along  $X$  and  $F^t v = \pi T f^t v$ . A flow  $f^t$  is *quasi-Anosov* if for  $v \in N$  the fact that  $|F^t v|$  is bounded for all  $t \in \mathbf{R}$  implies that  $v = 0$ . If  $\Lambda \subset M$  is invariant by a flow  $f^t$ , we say that  $\Lambda$  has a *hyperbolic structure* if there are  $0 < \lambda < 1$ ,  $C > 0$ , and a splitting  $TM|_{\Lambda} = E^u \oplus E^s \oplus \langle X \rangle$ , where  $\langle X \rangle$  is the bundle spanned by  $X = (df^t/dt)$ , such that for  $t \geq 0$

$$|T f^t v^s| \leq C \lambda^t |v^s| \quad \text{for } v^s \in E^s$$

$$|T f^{-t} v^u| \leq C \lambda^t |v^u| \quad \text{for } v^u \in E^u.$$

The condition for a set to have a hyperbolic structure for a diffeomorphism is the same but there is no bundle  $\langle X \rangle$ . A flow or diffeomorphism on  $M$  is *Anosov* if all of  $M$  has a hyperbolic structure. A flow *satisfies Axiom A* if the periodic points are dense in the nonwandering set,  $\Omega$ , and  $\Omega$  has a hyperbolic structure. See [3] or [8] for more definitions.

We define the stable and unstable bundles at all points  $x \in M$  by  $E_x^u = \{v \in T_x M : T f^t v \rightarrow 0 \text{ as } t \rightarrow -\infty\}$  and  $E_x^s = \{v \in T_x M : T f^t v \rightarrow 0 \text{ as } t \rightarrow \infty\}$ .