

# *On Algebraic Multiplicity*

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The purpose of the present paper is to improve, by sharpening, some parts of the spectral theory for nonselfadjoint operators. The new results include a fundamental duality between the algebraic multiplicities of a Fredholm operator  $T$  and its adjoint  $T^*$ , a decomposition criterion for non-Fredholm  $T$  and its relationship to, and use in, the Lyapunov-Schmidt alternatives method for reduction of nonlinear problems, an essential simplification concerning the two related questions of the coinciding of essential spectra and the so-called holding of Weyl's Theorem, and a new characterization of the Weyl-Browder limit point spectrum via the Weinstein-Aronszajn determinant theory of intermediate problems. These results are all obtained by algebraic multiplicity considerations.

Particular by-products are an answer to a question of Lay [18], some further clarification following Amelin [1] of a paper by Bouldin [3], and the correction of an error in a paper by Salinas [22]. A further sharpening is rendered by the inclusion of a number of examples and a specific counter-example construction that should be useful elsewhere.

An outline of the paper is as follows.

For  $A$  an operator in a Banach space and  $T = A - \lambda$ , an important index of algebraic multiplicity is the ascent of  $T$ , denoted by  $\text{asc}(T)$ , the smallest integer  $n \geq 0$  such that the operator null spaces  $N(T^n) = N(T^{n+1})$  agree. Finite algebraic and geometric multiplicities thus coincide if and only if  $\text{asc}(T) = 1$ , as in the selfadjoint case; in general the algebraic multiplicity may be defined as  $\dim N(T^{\text{asc}(T)})$ . A related index is the descent of  $T$ , denoted by  $\text{des}(T)$ , the smallest integer  $m \geq 0$  such that the operator ranges  $R(T^m) = R(T^{m+1})$  agree. In section 1 we make precise this relationship by establishing the duality  $\text{asc}(T) = \text{des}(T^*)$  for  $T$  a Fredholm operator; we also comment on the limitations of this duality beyond the class of Fredholm operators, in contrast to the well-known (geometric) index theory for operators. It is a little surprising that this basic duality has apparently gone unobserved, *e.g.*, see the recent survey article by Krachkovskii and Dikanskii [17].

In seeking nontrivial solutions to nonlinear eigenvalue problems of the form  $Tw + N(w) = \lambda w$ , where  $N$  is a nonlinear operator, the Lyapunov-Schmidt extension of the Fredholm alternatives method is often useful, especially in the