

Asymptotic Behavior of C_0 Semi-Groups as Determined by the Spectrum of the Generator

MARSHALL SLEMROD

1. Introduction. In applied mathematics one often desires to know the asymptotic behavior of a linear autonomous evolution equation in time. More specifically one may desire to know how fast solutions decay to the zero equilibrium solution. For systems of ordinary differential equations the results are well known and may be found in any standard text [1], [2]. The basic result is that for an ordinary differential equation $\dot{x} = Ax$, A an $n \times n$ constant matrix, $x(0) = x_0$, if the real part of spectrum of A , $\text{Re } \sigma(A)$, satisfies the condition $\text{Re } \sigma(A) \leq -\delta < 0$, then $|x(t)| \leq Me^{-\mu t} |x_0|$ for $t \geq 0$ for some $\mu > 0$, $M \geq 1$. One may pose a similar question in the setting of semigroups of bounded operators. Let $T(t)$ be a C_0 semigroup of bounded linear operators on a Banach space \mathfrak{X} with infinitesimal generator A . Let us assume $\text{Re } \sigma(A) \leq -\delta < 0$. What can one surmise as to the behavior of $T(t)x$, $x \in \mathfrak{X}$, as $t \rightarrow \infty$? The answer in general is nothing. The classical counter example of [3] provides us with a semigroup with $\sigma(A)$ empty such that $\|T(t)\| = e^{\pi^{1/2}t}$. Furthermore Zabczyk has shown the following result:

Theorem 1.1[4]. *For any two real numbers $\omega < \omega_0$ there exists a C_0 semigroup $T(t)$, $t \geq 0$, on a Hilbert space \mathfrak{X} such that $\sup \text{Re } \sigma(A) = \omega$ and $\|T(t)\| = e^{\omega_0 t}$.*

Thus without any restrictions on either the generator or the semigroup results similar to the classical finite dimensional result are impossible.

Results based on assuming properties of the semigroup are based on spectral theory. We know for a C_0 semigroup that

$$\|T(t)\| \leq M_\omega e^{\omega t} \quad \text{for all } \omega > \omega_0 \quad \text{where } \omega_0 = \lim_{t \rightarrow \infty} \frac{\log \|T(t)\|}{t}.$$

Also we know that $r(T(t)) = e^{\omega_0 t}$ (r denotes the spectral radius). Thus the value of ω_0 may be computed from the spectral radius of $T(t)$. Unfortunately the spectrum of $T(t)$ is not faithful to the spectrum of A , i.e. while $e^{P\sigma(A)t} \cup e^{R\sigma(A)t} \subset P\sigma(T(t)) \cup R\sigma(T(t)) \subset e^{P\sigma(A)t} \cup e^{R\sigma(A)t} \cup \{0\}$, for the troublesome continuous spectrum we know only $e^{iC\sigma(A)t} \subset C\sigma(T(t))$. Thus if we make some