

The Commutant of a Class of Analytic Toeplitz Operators II

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In this paper we characterize the commutants of certain analytic Toeplitz operators in terms of the well-known commutants of unilateral shifts. This is a continuation of the investigation begun in [10], [11], and we refer the reader to those papers for notation, terminology, and historical background of the problem. The problem under consideration is finding the most general φ for which the commutant $\{T_\varphi\}'$ of the analytic Toeplitz operator T_φ is equal to $\{T_B\}'$ for some inner function B of which φ is a function. An example of Abrahamse [12] shows that some assumption on φ is necessary. The main result of [10] is that φ analytic in \bar{D} is a sufficient condition.

Theorem. *Let φ be in H^∞ and suppose that the inner part of $\varphi - \varphi(\lambda)$ is a finite Blaschke product for uncountably many λ in D . Then there exist an H^∞ function ψ and a finite Blaschke product B such that $\varphi(z) = \psi(B(z))$ and $\{T_\varphi\}' = \{T_B\}'$.*

Because of the length of the proof of the theorem, we place it at the end of the paper.

Corollary 1. *Let φ be in H^∞ . Then $\{T_\varphi\}' = \{T_B\}'$ where B is a finite Blaschke product of which φ is a function, if either of the following hold:*

- (a) *The function φ is p -valent for some positive integer p .*
- (b) *The function φ extends continuously to \bar{D} , and there exists λ in D such that $\varphi(\lambda) \notin \varphi(\partial D)$. In particular, the latter condition holds if $\varphi(\partial D)$ has planar measure zero.*

Proof. (a) Let $q(z)$ be the number of points in $\varphi^{-1}(\varphi(z))$, and let M be the open subset of D on which q attains its maximum. If $z \in M$, then $\varphi - \varphi(z)$ is bounded away from zero near ∂D , and thus the inner part of $\varphi - \varphi(z)$ is a finite Blaschke product. Applying the theorem, we obtain the desired conclusion.

(b) If $\varphi(\lambda) \notin \varphi(\partial D)$, then $\varphi(z) \notin \varphi(\partial D)$ for every z in some neighborhood N of λ . Thus the inner part of $\varphi - \varphi(z)$ is a finite Blaschke product for every z in N , and the theorem implies the desired conclusion.

The set of functions in H^∞ that extend continuously to \bar{D} is called the disk