

Parabolic Boundary Layers

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1. Introduction. In this paper we study parabolic boundary layers for semilinear elliptic and parabolic partial differential equations in two independent variables. The Dirichlet problem is treated for elliptic equations and the first boundary value problem for parabolic equations. Our method is probabilistic depending upon the representation of the solution as the expected value of a functional of an Ito stochastic differential equation.

Parabolic boundary layers for linear elliptic equations were first treated by Kamenomostskaya [5] in the special case the entire boundary is tangent to characteristics of the reduced equation. Eckhaus and deJager [1] gave the first comprehensive treatment of linear elliptic equations. Grassman [2] and Mauss [6] considered special cases and obtained more detailed results.

Using our probabilistic method we obtain results which depend only on local estimates for the region involved. In addition, it allows us to treat a class of semilinear equations provided a compatibility condition is satisfied (see Theorem 2). Finally, by avoiding the maximum principle approach of Eckhaus and de Jager [1], we are able to allow the coefficient $m(x)$ (see equation (9)) to be arbitrary rather than strictly negative as required in [1]. It should be remarked, however, that for special problems the work in [1], [2], and [6] contains results which are not covered here.

Parabolic boundary layers for linear parabolic equations were briefly treated by Kamenomostskaya [5]. The results there for the linear case are weaker than those presented here. In particular, the effect of the compatibility condition ensuring that the solution W^0 (see equations (11), (12)) is C^2 in \bar{S} is not discussed.

II. The elliptic case. Let $z = (z_1, z_2) \in \mathbf{R}^2$. For $\epsilon > 0$ consider the elliptic equation

$$(1) \quad \epsilon(\bar{a}(z)u_{z_1 z_1} + 2\bar{b}(z)u_{z_1 z_2} + \bar{c}(z)u_{z_2 z_2}) + \bar{f}(z)u_{z_1} + \bar{g}(z)u_{z_2} + \bar{F}(z, u) = 0,$$

in B an open subset of \mathbf{R}^2 , with boundary data $u = \Lambda$ on ∂B , the boundary of B . Parabolic boundary layers occur when the characteristics of the reduced equation with $\epsilon = 0$ are tangent to a portion of ∂B . Suppose that there exists a subset D of $B \cup \partial B$ with the following properties:

(i) there exist functions l_1, l_2 defined on $[0, 1]$ of class C^∞ such that the mapping $s \rightarrow (l_1(s), l_2(s))$ is one-to-one with $(l_1(s), l_2(s)) \in \partial B$. Define $\ell = (l_1, l_2)$.