

Interior Gradient Bounds for Non-uniformly Elliptic Equations

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In [1] Bombieri, De Giorgi and Miranda were able to derive a local interior gradient bound for solutions of the minimal surface equation with n independent variables, $n \geq 2$, thus extending the result previously established by Finn [2] for the case $n = 2$. Their method was to use test function arguments together with a Sobolev inequality on the graph of the solution (Lemma 1 of [1]). A much simplified proof of their result was later given by Trudinger in [12].

Since the essential features of the test function arguments given in [1] generalized without much difficulty to many other non-uniformly elliptic equations, it was apparent that interior gradient bounds could be obtained for these other equations provided appropriate analogues for the Sobolev inequality of [1] could be established. Ladyzhenskaya and Ural'tseva obtained such inequalities ([4], Lemma 1) for a rather large class of equations, including the minimal surface equation as a special case. They were thus able to obtain gradient bounds for this class of equations.

In §2 of [5] a general Sobolev inequality was established on certain generalized submanifolds of Euclidean space. In the special case of nonparametric hypersurfaces in \mathbf{R}^{n+1} of the form $x_{n+1} = u(x)$, where u is a C^2 function defined on an open subset $\Omega \subset \mathbf{R}^n$, the inequality of [5] implies

$$(1) \quad \left\{ \int_{\Omega} h^{n/(n-1)} v \, dx \right\}^{(n-1)/n} \leq c \int_{\Omega} \left[\left\{ \sum_{i,j=1}^n g^{ij} h_{x_i} h_{x_j} \right\}^{1/2} + h |H| \right] v \, dx$$

for each non-negative C^1 function h with compact support in Ω , where

$$\begin{aligned} v &= (1 + |Du|^2)^{1/2} \\ g^{ij} &= \delta_{ij} - v_i v_j, \quad v_i = u_{x_i}/v, \quad i, j = 1, \dots, n \\ H &= \frac{1}{n} v^{-1} \sum_{i,j=1}^n g^{ij} u_{x_i x_j}, \end{aligned}$$

and where c is a constant depending only on n . (See the discussion in §2 below.) The quantity H appearing in this inequality is in fact the mean curvature of the hypersurface $x_{n+1} = u(x)$ and in the special case when $H \equiv 0$ (*i.e.* when u