

A Differential Fundamental Theorem of Algebra

ROBERT W. WEST

§1. Introduction. In order to state our main result, we need some terminology. Let M be a (real) manifold of class C^r , $r \geq 0$, and let $A \subset M$ be any subset. By a *monic polynomial function on A of degree n and class C^r* we mean a function $F : A \rightarrow \mathbf{C}[t]$, t an indeterminate, of the form

$$F(x) = f(t, x) = t^n + a_1(x)t^{n-1} + \cdots + a_n(x)$$

where $a_i : A \rightarrow \mathbf{C} \equiv \mathbf{R}^2$ is a map of class C^r , $i = 1, \dots, n$. (If A is not open, this means that each a_i extends to a C^r map on some open neighborhood of A in M .) We use the capital letter F to draw attention to the function, and its counterpart f is used when we need to do something with the indeterminate t . A function $r : A \rightarrow \mathbf{C}$ is a *root function of F* if $r(x)$ is a root of $F(x)$ for each $x \in A$, i.e. $f(r(x), x) = 0$ for all x .

Our main result is the following Differentiable Fundamental Theorem of Algebra:

(1.1) Theorem. *Let M be a differentiable manifold of class C^r , $r \geq 1$, and let $F : M \rightarrow \mathbf{C}[t]$ be a monic polynomial function of degree $n \geq 1$ and class C^r . Let B be the set of points $x \in M$ for which the polynomial $F(x)$ has multiple roots, and assume that*

(i) *if $\dim M > 1$, then B is a closed discrete subset of M and $M - B$ is simply connected;*

(ii) *if $\dim M = 1$, then B is finite and M is simply connected. Then there exist continuous root functions $r_i : M \rightarrow \mathbf{C}$, $1 \leq i \leq n$, such that*

$$f(t, x) = (t - r_1(x))(t - r_2(x)) \cdots (t - r_n(x)) \in \mathbf{C}[t]$$

for all $x \in M$. Moreover, if $r_i(x)$ is a simple root of $F(x)$, then r_i is of class C^r at point x . In particular, r_i is of class C^r on $M - B$.

Note that B may be empty in the theorem.

The following examples give some insight into the technical aspects of the above theorem.

(1.2) Example. Let S^1 consist of complex numbers $z = e^{i\alpha}$, $\alpha \in \mathbf{R}$, and set