

Spectral Properties of an Acoustic Boundary Condition

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1. Introduction. In this paper we study a boundary condition for the wave equation occurring in theoretical acoustics. The initial value problem in a bounded domain is solved by semigroup methods in a Hilbert space of data with finite energy. A description of the spectrum of the semigroup generator A is then obtained. Unlike the generators associated with the usual boundary conditions, which have compact resolvent and spectrum consisting of discrete eigenvalues, A always has essential spectrum. Moreover, if the parameters occurring in the boundary condition are constant, there are sequences of eigenvalues converging to the essential spectrum. The distinction has to do with the fact that the equation $(\lambda - A)\mathbf{u} = \mathbf{f}$ reduces to a boundary value problem in which the boundary condition depends explicitly on λ .

The problem arises as follows. Suppose G is a bounded domain in \mathbf{R}^3 filled with a fluid which is at rest except for acoustic wave motion. If $\phi(x, t)$ is the velocity potential, so that $-\nabla\phi(x, t)$ is the particle velocity, then ϕ satisfies the wave equation

$$(1.1) \quad \phi_{tt} = c^2 \Delta\phi \text{ in } G,$$

where c is the speed of sound in the medium. Now suppose that ∂G is not rigid but subject to small oscillations. We assume that each point reacts to the excess pressure of the acoustic wave like a resistive harmonic oscillator. We also assume the different parts of the boundary do not influence each other; such a surface is called *locally reacting*. Then the normal displacement δ of the boundary into the domain satisfies an equation of the form

$$(1.2) \quad m(x)\delta_{tt}(x, t) + d(x)\delta_t(x, t) + k(x)\delta(x, t) = -\rho\phi_t(x, t) \text{ on } \partial G,$$

where ρ is the density of the fluid. If we also assume the boundary is impenetrable, we obtain a third equation from the continuity of velocity at the boundary,

$$(1.3) \quad \delta_t(x, t) = \phi_n(x, t) \text{ on } \partial G$$

where ϕ_n is the outward normal velocity. This model is used for example in [10] for waves assumed to be at a definite frequency (see p. 263).