

# *Elementary Critical Point Theory and Perturbations of Elliptic Boundary Value Problems at Resonance*

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**1. Introduction and summary.** Basic references for the standard notation and terminology used in this section are [1] and [5].

Let  $\Omega$  be a bounded domain in  $\mathbf{R}^n$  ( $n \geq 1$ ) and suppose

$$Lu = \sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq 1}} (-1)^{|\alpha|} D^\alpha a_{\alpha\beta} D^\beta u$$

is a uniformly elliptic, formally self adjoint second order operator defined on  $\Omega$  with real coefficients  $a_{\alpha\beta} \in L^\infty(\Omega)$ . Let  $f : \mathbf{R} \times \bar{\Omega} \rightarrow \mathbf{R}$  be continuous and bounded. We are concerned with the existence of weak solutions of the problem

$$(1) \quad \begin{aligned} (Lu)(x) &= f(u(x), x) & x \in \Omega \\ u(x) &= 0 & x \in \partial\Omega. \end{aligned}$$

If the linear homogeneous problem

$$(2) \quad (Lu)(x) = 0 \quad \text{on } x \in \Omega, \quad u(x) = 0 \quad x \in \partial\Omega$$

has only the trivial weak solution, then the solvability of (1) follows from a straight-forward application of Schauder-Leray theory. The interesting case, therefore, is when the problem (2) has nontrivial solutions (the *resonance case*).

Before stating our main results we make the notion of a weak solution of the nonlinear problem more precise.

Let  $\dot{H}_1$  denote the usual real Sololev space formed by completion of the real inner product space of  $C^1$  functions with compact support contained in  $\Omega$  and inner product

$$(3) \quad \langle u, v \rangle_1 = \langle u, v \rangle_0 + \sum_{|\alpha|=1} \langle D^\alpha u, D^\alpha v \rangle_0$$

where  $\langle \cdot \rangle_0$  is the usual real  $L^2(\Omega)$  inner product. Let

$$(4) \quad B(u, v) = \sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq 1}} \langle a_{\alpha\beta} D^\alpha u, D^\beta v \rangle_0$$